On Bolzano's Concept of a Sum

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Introduction

Well known for his work in logic and the philosophy of mathematics, Bolzano also made significant contributions to ontology, most notably in his theory of collections. Though recent work has done much to illuminate Bolzano's conceptions,[1] modern readers have still encountered a variety of difficulties in understanding them. Nowhere have these been so pronounced as with his conception of the special kind of collections he calls *sums*. For example, Frank Krickel, who has written the most detailed study of Bolzano's theory of collections, wrote that Bolzano's concept of a sum

> remains plagued with an entire series of unclarities; indeed, I might even claim that among all the unclear concepts of Bolzano's theory of collections that of a sum remains the hardest to understand precisely.[2]

With respect to Bolzano's *exposition*, I shall remain in full agreement with Krickel. It is deficient to the point where readers cannot reasonably be expected to understand what Bolzano means by a sum. By contrast, I shall attempt to show that Bolzano's *concept* of a sum is not only unexceptionable but also rather interesting.

On collections in general

Although the most detailed presentation of Bolzano's theory of collections remained unpublished until late in the twentieth century,[3] the theory was sketched in the *Theory of Science* and the *Paradoxes of the Infinite*,[4] and was known to the pioneers of both set theory and mereology. There is no mistaking the fact that Bolzano's conceptions have common points with those of later authors such as Cantor and Leśniewski. All the same, attempts to interpret Bolzano's collections as sets or mereological wholes in the modern sense face insuperable difficulties, and I think Peter Simons was exactly right in deeming Bolzano's theory to be *sui generis*.[5]

[1] *Sebestik 1992; Krickel 1995; Simons 1997; Behboud 1997* are particularly helpful.
[2] *Krickel 1995*, p. 99.
[3] *Bolzano 1975*.
[4] *Bolzano 1837*, §§82–87; *Bolzano 1851*, §§3–10.
[5] *Simons 1997*, p. 87.

For Bolzano a collection [*Inbegriff*], is a whole with parts or, as he prefers to say, something that has complexity.[6] Thus, in marked contrast to what we find in set theory, there are no empty collections and no singletons. Another singularity of Bolzano's conception is his requirement that no part of a collection be part of any other part of the same collection.[7] This prohibition is a special case of a more general one to the effect that no object may occur repeatedly within the same collection. Thus, for example, if the ideas $[A]$ and $[D]$ represent the same object, the idea of a collection whose parts are $A, B, C, D, \ldots$ has no object.[8] His reason for this is that 'it is obviously impossible for something to be united *with itself* to form a collection, i.e., a new object distinct from itself.'[9] Thus if A and B are parts of a collection, then we must have not only $A \neq B$ but also A must not be a part of B, and conversely. Objects familiar from set theory where the same object figures repeatedly, such as the Kuratowski ordered pair $\{\{a\}, \{a, b\}\}$, would thus not seem to cut it as collections in Bolzano's sense. This *no-redundancy* principle, as Simons called it, seems to be the only one Bolzano thinks necessary for the existence of collections, since he claims at one point that any distinct objects form a collection, provided only that none of them is a part of any of the others.[10]

Some collections are structured, and some are not. For example, a collection whose parts are the points between two given points might or might not be considered as having a certain structure, e.g., that determined by the relation of betweenness. Thus simply indicating the parts of a collection may not be enough to identify it, and one cannot speak of *the* collection containing given objects, since in general there are many different collections with the same parts.[11] Being endowed with a certain structure is an optional feature, however, and by no means follows from the mere concept of a collection:

> [I]n the mere idea of a collection it is not presupposed that the order or sequence in which the united objects appear is fixed, nor even that there is or even can be such an order.[12]

[6]*Bolzano 1975*, §6 (p. 100); though the sense in which Bolzano takes the word *Inbegriff* has not survived in modern German, it was accepted usage in his day. Adelung (1808) gives only two senses of '*Inbegriff*': 1) an enclosed area, and 2) a collection of things enclosed within certain limits, where enclosure is often taken in a figurative sense. The first sense, he adds, is hardly used anymore. In the late nineteenth and early twentieth centuries, one finds '*Inbegriff*' used in a sense similar to Bolzano's by Cantor (*1962*, p. 115), Husserl (*2007*, passim), and Dedekind (*1932*, p. 345), among others. Hilbert and Bernays (*1992*, p. 97) still used the word in a similar sense as late as 1919-20.

[7]*Bolzano 1975*, §6 (p. 101).

[8]*Bolzano 1975*, §14 (p. 105).

[9]*Bolzano 1975*, §14 (p. 105).

[10]*Simons 1997*, p. 98; *Bolzano 1975*, §6 (p. 101).

[11]*Bolzano 1975*, §12 (p. 105).

[12]*Bolzano 1837*, §82 [I.393].

Note, too, that concepts are involved in determining what a given collection is.[13] The concept of a given collection determines not only whether the ordering of parts matters for its individuation, but also what its parts are. This latter feature becomes especially salient when Bolzano comes to discuss what he calls *pluralities* [*Vielheiten*].[14] A concept such as [grapefruit], for example, picks out certain objects. Though these objects are complex, containing other parts (peel, rind, seeds, segments, cells, molecules, atoms, electrons, etc.), when we consider a plurality *of grapefruits*, individual grapefruits are *considered* simple as parts of that particular collection.[15] I shall speak in such cases of the *relatively simple* parts of a collection. Thus an individual grapefruit is considered simple with respect to a given plurality of grapefruits (e.g., a dozen), even though in other respects it is a highly complex object.

Multitudes [*Mengen*]

In the notion of a collection as such, as noted, no mention of the arrangement of parts occurs. In some collections, arrangement or structure matters (e.g., a building as a structured collection of beams, girders, pipes, wires, wallboard, screws, etc.); in others, it does not (e.g., a quantity of ore, where we are interested only in how much iron it contains). In either case, this is something that has to be specified in the concept of the collection. If the arrangement of parts is held to be a matter of indifference, Bolzano calls the collection a *Menge*. Put otherwise, a *Menge* is a collection that retains its identity under all rearrangements of its parts.

Menge is, of course, the word that is used in modern German for sets, and Bolzano's *Mengen* do share many features with sets in the modern sense. For example, he appears to think that a whole is completely determined by its parts and the way these are combined (the structure of the whole). Since in *Mengen*, the arrangement of parts is a matter of indifference, these collections satisfy a principle of extensionality, which he states as follows:

> The parts of which a *Menge* consists determine it, and this completely. . . .[16]

Thus it does make sense to speak of the *Menge* containing given objects, provided that there is such a collection (i.e., provided that the objects satisfy the no-redundancy condition).

[13]This does not mean that collections are mind-dependent in any way. For Bolzano maintains that, in addition to thought or expressed concepts, there are also what he calls *concepts in themselves*. It is the latter that are involved in the existence and nature of collections. See, e.g., *Bolzano 1975*, §6 (p. 100–101).

[14]*Bolzano 1975*, §119 ff (p. 166 ff).

[15]*Bolzano 1975*, §120 (p. 169).

[16]*Bolzano 1975*, §89 (p. 152).

But anyone who identifies Bolzano's *Mengen* with modern sets, as Simons has observed, does so to his own peril.[17] This is perhaps most obvious in the case of the special kind of *Mengen* Bolzano calls *sums*.[18] For example, Bolzano represents a sum with three (relatively) simple parts A, B, C as follows:

$$A + B + C$$

and he tells us[19] that it is of the very nature of a sum that:

$$A + B + C = A + (B + C)$$

Any attempt to provide a direct translation of Bolzano's notation into that of set-theory immediately encounters serious problems in the face of such claims. If, for example, $A+B+C$ is identified with the set $\{A, B, C\}$, what set is represented by $A + (B + C)$? The first thing that might occur to us is $A + (B + C) = \{A, \{B, C\}\}$. But then Bolzano's claim that with sums $A+B+C = A+(B+C)$ would come out plainly false, since the sets $\{A, B, C\}$ and $\{A, \{B, C\}\}$, having different elements, are themselves distinct. Was he simply confused, using 'part' to represent both set-membership and the subset relation, so that both B and $(B+C)$ could be said to be parts of the sum, but in different senses? If so, his point might have been that because $\{B, C\} \subset \{A, B, C\}$ and $B, C \in \{B, C\}$, we also have $B, C \in \{A, B, C\}$? But how could this be if sums are supposed to be a *special* kind of *Menge*—for the stated property holds universally for sets?

We shall see later (section 6) that, as one might expect, it is possible to construct set-theoretical *Ersätze* for Bolzanian sums. But it will also come as no surprise that this will not be accomplished by means of a direct translation of Bolzano's terminology and notation into that of set-theory.

Given the significant differences between Bolzano's *Mengen* and modern sets, Simons suggested the term 'multitude' as an English equivalent for Bolzano's '*Menge*', one not encumbered with the misleading associations of 'set'.[20] This proposal seems to have caught on, and in what follows, I shall follow it. By contrast, although the term 'sum' carries some unhelpful associations, I shall continue to use it for Bolzano's '*Summe*', as there seems to be no reasonable alternative.

Sums

What, then, is a sum? Sums, like multitudes in general, we are told, retain their identity under rearrangements or permutations of their parts. But they also have a

[17] *Simons 1997*, p. 87ff.

[18] In *Bolzano 1975* (§98, p. 157), we are told explicitly that sums are *Mengen*.

[19] *Bolzano 1851*, §5.

[20] *Simons 1997*, p. 95–96.

further property. The first characterization of this property in the *Theory of Science* occurs even before Bolzano has introduced the term 'sum':

> It is true only of collections of a *special* type that 'parts of a part are parts of the whole,' as one says. . . .[21]

This sounds like a transitivity condition on the part relation, and for the most part it has been interpreted as one. Jan Berg, for example, offers the following:

> If F contains x as a part, it also contains the parts of x as parts....[22]

So too Frank Krickel:

> The only thing that seems to be certain with collections for which the Sum-principle is valid is that the parts of parts are supposed to be parts of the whole. [23]

Peter Simons, by contrast, introduces a refinement, based on the remarks Bolzano makes just before the above citation. There, speaking of a collection that contains the three persons Caius, Sempronius, and Titus, Bolzano writes:

> It might well appear that the kind of ideas to which the above example belongs has been incorrectly grasped by the given definition. For we could call any limb on any of their bodies, or any collection composed of two of them, e.g., Titus and Sempronius, a part of the collection. On the other hand, when we say that each of these persons is an individual capable of filling a certain post, we do not want to say that the arms of Caius, or the two persons Caius and Titus together, are suitable for this office. It seems therefore that the two expressions 'Each of the objects $A, B, C, D, \ldots$' and 'Each part of the collection $A, B, C, D, \ldots$' do not have the same sense. This objection vanishes once we realize that according to the definition of 'part' that I gave in no. 1, only the objects of which we think as constituting a collection are to be envisaged as parts of that collection. We see, furthermore, that according to this definition things that are merely parts of these parts, likewise those that are themselves collections of such parts, have no claim to this name.[24]

[21] *Bolzano 1837*, §83, no. 2 [I.397].

[22] *Berg 1962*, p. 89; cf. *Berg 1985*, p. 16, *Berg 1992*, p. 34.

[23] *Krickel 1995*, p. 142. Lapointe (*2011*), who only touches on sums in passing (p. 118-119), seems to have had something similar in mind, though she also claims that the parts of parts must be of the same kind as the parts.

[24] *Bolzano 1837*, §83 [I.397].

As noted above, what counts as a part of a given collection is determined by the concept of that collection. Thus even though a given object is complex, it may still be considered relatively simple as a part of a given collection. Simons suggests that we call the objects that are considered genuine parts of a collection according to its concept its *components*, reserving the term ‘part’ for a more general sense in which, e.g., the arms of Caius are parts (i.e., of Caius), even though they are not components of the collection mentioned above. Based on the citation above, he then offers the following interpretation of the characteristic property of sums:

> If ... we have a multitude where every part of a component is a component, then Bolzano calls this a *sum*.[25]

Ali Behboud pointed out that this interpretation did not work for one of Bolzano’s more interesting examples, a line considered with respect to its length. Here, Behboud notes, even though Bolzano held that the ultimate parts of every line are points, points are not considered components of the sum in question.[26] Later, after observing that different kinds of parts might be distinguished (e.g., person-parts, body part-parts, arbitrary parts), he offers the following proposal in a conjectural spirit:

> We could try ‘u-w-transitivity’ defined by
>
> $$\forall x, y, z((x <_u y \& y <_w z) \to x <_w z)$$
>
> for any given pair of ideas u and w.[27]

In my opinion, none of these interpretations will work. Indeed, it seems to me that any attempt to interpret the characteristic property of sums as a transitivity condition on the part relation will have the unfortunate consequence that there are no sums.

Consider, to begin with, Simons’ interpretation: ‘If we have a multitude where every part of a component is a component, then Bolzano calls this a sum,’ i.e., if z is to be a sum, we must have:

$$\forall x, y((Pxy \wedge Cyz) \to Cxz) \tag{1}$$

Now recall the no-redundancy condition, which Simons states for sums as follows:

[25] *Simons 1997*, p. 96. In *Simons 2000,* p. 221, he gives a different formulation, which is closer to those of Berg and Krickel: ‘Unter den Mengen gibt es solche, bei denen sämtliche Teile eines Teiles selbst Teile dieser Mengen sind. Bolzano nennt sie *Summen*.’

[26] *Behboud 1997*, p. 111; this example is discussed further below (section 6).

[27] *Behboud 1997*, p. 114 note 13.

> The same desire to avoid redundancy leads Bolzano to require in the case of sums that no components share a part. This is derivable from the more general requirement that no component have another as a part....[28]

That is,

$$\forall x, y((Cxz \wedge Cyz) \rightarrow \neg Pxy)$$

But it follows from these two principles that the antecedent of condition (1) is never satisfied in the case of sums, i.e.:

$$\neg\exists x, y(Pxy \wedge Cyz)$$

which would mean that the characteristic condition distinguishing sums from multitudes could only be satisfied vacuously. But since Bolzano holds that universal propositions are false if their subject-concepts are empty, it would follow that there are no sums, against his insistence that there are such collections.

It is clear that the formulations of Berg and Krickel, which do not distinguish parts and components, are open to a similar objection.

Behboud's generalized proposal may seem to offer a way out, in that one might maintain that even if x is a v-part of y and y a w-part of z, x could still be a w-part of z, provided that it was not a w-part of y. The no-redundancy condition might then be preserved with respect to w-parts. But this seems to be in disaccord with Bolzano's motivation for introducing the no-redundancy condition in the first place: if x has already been incorporated into z as any sort of part of y, what sense can there be in talk of adding x to y, possibly along with other parts, to obtain the whole z? Nor would it sit well with his examples. When, for example, a line segment that is part of a larger line segment is replaced by its two halves, the latter seem to be parts of the former in exactly the same sense as they are parts of the whole. Or consider again Bolzano's claim that it lies in the very concept of a sum that:

$$A + (B + C) = A + B + C$$

Is it not the case that B is a part of $(B + C)$ in the very same sense that it is a part of $A + B + C$? If so, then the option of distinguishing different kinds of parts is not open to us.

In brief, transitivity of the part relation, a standard feature of later theories of whole and part, is incompatible with Bolzano's other commitments, not to mention his explicit statement that no part of any collection can be a part of another part.

So let us ask again: what is a sum? Let us begin by noting that the first formulation is not Bolzano's own, but rather a report of a common manner of speaking:

[28] *Simons 1997*, p. 98.

> It is true only of collections of a *special* type that 'parts of a part are parts of the whole,' *as one says*....
>
> *Was man zu sagen pflegt,* daß die Theile eines Theiles auch Theile des Ganzen wären, gilt nur bei Inbegriffen einer gewissen Art.... [29]

For his part, Bolzano sometimes puts things differently:

> I allow myself to call *sums* those collections in which the manner of combination [of the parts] does not matter, and in which the parts of the parts may be looked upon as parts of the whole.[30]

Note that he no longer says that the parts of parts *are* parts of the whole; rather, they *may be regarded* as such. Thus transitivity does not seem to be at issue. Immediately afterwards, he states the following theorem:

> It follows from the concept of a sum that it is not changed if the order of its parts is changed, and that it is not changed if one of its parts is *replaced by* the parts of that part.[31]

We now have something we can work with. Recall that multitudes were characterized as collections that retain their identity under certain kinds of transformations, namely, permutations or rearrangements of their parts. Sums, similarly, may be regarded as collections that retain their identity under the rearrangement of their parts *and* when one of their parts is replaced by the parts of that part.

Consider again one of the simplest examples, a sum of three (relatively simple) objects A, B, C.[32] Bolzano says that:

$$A + (B + C) = A + B + C$$

That is, if we replace the part $(B + C)$ by its parts, the sum retains its identity. B and C, as parts of parts of the sum $A + (B + C)$ *may be considered* parts of the same sum—but this will be *with respect to a different partition.* One and the same sum, that is, may be partitioned in a variety of ways. A remark in the *Paradoxes of the Infinite* provides further support for this reading. There, Bolzano says of a given collection that it is 'not considered as a sum, and thus not divisible into

[29] *Bolzano 1837*, §83, no. 2 [I.397].

[30] *Bolzano 1837*, §84 [I.400]; underlining added.

[31] *Bolzano 1837*, §84 [I.400]; emphasis added. Cf. *Bolzano 1975* §93 (p. 154): '. . . nothing changes in a sum when one rearranges the parts of which it is composed in any manner whatsoever, nor when, in place of all or some of these parts, one introduces the parts of which these themselves consist.'

[32] *Bolzano 1851*, §5.

arbitrary multitudes of parts';[33] by contraposition, to look upon a whole as thus divisible is to think of it as a sum.

Jan Sebestik may have had something like this in mind when he wrote:

> The defining property, 'the parts of parts are parts of the whole', in fact translates into the invariance of the sum with respect to the substitution of remote parts (parts of parts) for proximal parts. Whether it be the content of a concept, an arithmetical sum, or a sum of quantities (the length of a line), the sum remains the same even if one substitutes parts of parts for the parts of the whole. A sum is thus a 'quasi-set' that is invariant with respect to any decomposition of its elements. Sums are 'sets' where one may take as a part (element) the result of any partition of any part.[34]

But there seems to be more to sums than this. For identity is symmetrical, so if a sum retains its identity when a given part is replaced by its proximal parts, it stands to reason that the same will hold for the inverse procedure. That is, if certain kinds of disaggregation or dissolution do not affect the identity of a sum, neither will the corresponding kinds of *aggregation* or *fusion*. Thus just as the operation that transforms $A + (B + C)$ into $A + B + C$ leaves the sum of A, B, C invariant, so too will the operation transforming $A + B + C$ into $A + (B + C)$.

I think it likely that Bolzano believed that the second feature (invariance under certain kinds of aggregation or fusion) was an immediate consequence of the first, given the symmetry of identity. Recall, for example, that just before first touching on sums he says that with respect to the parts of collections in general, as opposed to the special kind he shall speak of later (i.e., sums),

> ...things that are merely parts of these parts, *likewise those that are themselves collections of such parts*, have no claim to this name [*sc.*, 'part'].[35]

A remark that strongly suggests that he thought invariance under aggregation was also built into the concept of a sum.

On my interpretation, then, sums are collections that retain their identity under three kinds of transformations:

1. Rearrangement/permutation of parts (it is this feature that led Bolzano to claim that sums are multitudes)

[33] *Bolzano 1851*, §33.
[34] *Sebestik 1992*, p. 321–322.
[35] *Bolzano 1837*, §83, no. 2 [I.397]; emphasis added.

2. Disaggregation/dissolution of proximal parts into their proximal parts

3. Aggregation/fusion of certain proximal parts.

An example may be helpful at this point. Bolzano represents a whole with parts A, B, C as (A, B, C). If one of these parts, say A, is itself a whole composed of parts p, q, r, he will write $([p, q, r], B, C)$, etc. Suppose now that we have a collection C, not a sum, which is partitioned and sub-partitioned as indicated below:

$$([(a, b), c], d, [e, f, g])$$

where a, b, c, d, e, f, g are (relatively) simple.

If we now think of the same whole but make the additional stipulation that the operations of rearrangement, disaggregation, and aggregation will not change the identity of the collection, we have the concept of a sum C'. Thus, to show but a few of the possible transformations, $C' =$

$$= ([(a, b), c], d, [e, f, g]) \qquad (1)$$
$$= ([(a, b), c], [e, f, g], d) \qquad (2)$$
$$= (d, [(a, b), c], [e, f, g]) \qquad (3)$$
$$= (d, (a, b), c, [e, f, g]) \qquad (4)$$
$$= (d, a, b, c, [e, f, g]) \qquad (5)$$
$$= (d, a, b, c, e, f, g) \qquad (6)$$
$$= (a, b, e, f, g, c, d) \qquad (7)$$
$$= ([a, b], [e, f], g, [c, d]) \qquad (8)$$

and so on,

where all of the above expressions indicate what one might call *aspects* of one and the same sum C'. (Here, 1 is the original partition, 2–4 result from the previous item by rearrangement, 5–7 by disaggregation and 8 by aggregation.)

Notice that, in general, there is no uniquely determined answer to either of the questions: *How many parts does a given sum have?* and *What are its parts?* For if

$$C' = ([(a, b), c], [e, f, g], d) = (a, b, c, d, e, f, g)$$

then the sum has either three or seven parts, depending on how we look at it. So, too, b is a part of C' relative to the latter but not the former partition. Bolzano's point is that it remains the same sum, regardless of how it is partitioned.[36] We thus have no absolute notion of *part* in the case of sums, but rather only the relative one

[36] As usual, this partitioning will be subject to constraints in most cases, determined by the concept of the sum in question. A plurality of grapefruits, for example, will not admit of further partitioning once we reach the individual grapefruits.

of a part *with respect to a given partition* or *aspect*. From this we can of course obtain the absolute notion: *part with respect to some partition or other*. With *this* absolute notion of a part of a sum (which Bolzano did not himself explicitly adopt), moreover, we *would* have transitivity of the part relation, as in modern part-whole theories.

This interpretation may also help to explain why Bolzano thought that the parts of a sum completely determine it.[37] Namely, given any one aspect of a sum, with its determinate divisions and subdivisions, and the stipulation that disaggregation, rearrangement, and aggregation are permitted, we can obtain any other aspect. The whole, along with all of its partitions, is thus determined by any one of them.

Bolzano's examples of sums

Bolzano gives a variety of examples of sums, among them:

- The content of an idea as the sum of its parts.[38]
- Pluralities [*Vielheiten*];[39] hence concrete numbers (a pair of aces, a trio of musicians, a covey of pheasants, a bevy of beauties, etc.)
- Arithmetical sums, e.g., 7+5 [40]
- Totalities, e.g., the universe (the sum of all substances whose existence has a ground)[41]
- A line, with respect to its length[42]
- The sum of an oven, a clock and a painting[43]
- The sum of 3 cubic inches, 5 days and 2 hundredweight[44]

Several of these examples deserve further comment.

Content of an idea To begin with, Bolzano's decision to call the sum of the parts of an idea its content reflects his belief that when we consider the content of an idea, we think simply about which parts it has, not about how they are combined

[37] *Bolzano 1975*, §98 (p. 157).
[38] *Bolzano 1837*, §56.
[39] *Bolzano 1975*, §119; in *Bolzano 1837*, §86 pluralities are not explicitly claimed to be sums.
[40] *Bolzano 1837*, §84, note [I.400-401].
[41] *Bolzano 1975*, §119 (p. 167).
[42] *Bolzano 1837*, §84 [I.400].
[43] *Bolzano 1975*, §93 (p. 154).
[44] *Bolzano 1975*, §93 (p. 155).

(their *Verbindungsart*). Thus, for example, the ideas [learned son of an ignorant father] and [ignorant son of a learned father] have the same content, differing only in the arrangement of the parts. So too the ideas $[3^5]$ and $[5^3]$. We see, too, how in each of his examples, a sequence of transformations involving disagreggation, rearrangement, and aggregation can produce each idea of the pair from the other. Note that the content of an idea in this sense differs from the multitude (set) of its simple parts. For the concept of a sum allows us, for example, to regard [learned father] as well as [father] as belonging to the content of the idea [ignorant son of a learned father], both being parts of the sum (with respect to different partitions). This example also makes it clear that invariance under aggregation, and not merely under disaggregation, must be a feature of sums, since if the ideas [learned son of an ignorant father] and [ignorant son of a learned father] have the same content, and [learned son] belongs to the content of the former idea, it must also belong to the content of the latter.

Pluralities and arithmetical sums The case of pluralities and arithmetical sums is particularly important for Bolzano's work on the foundations of arithmetic. He tells us repeatedly that the commutativity and associativity of addition are built into the very concept of a sum. We are now in a position to see what he had in mind. If $7 + 5$ and $(2 + 3) + 5$, for example, are sums in Bolzano's sense, then, since rearrangement of parts does not affect the identity of sums, we have immediately that $7+5 = 5+7$ and $(2+3)+5 = 5+(2+3)$. And since disaggregation and aggregation do not alter a sum, we also have $(2+3)+5 = 2+3+5 = 2+(3+5)$.

Even more interesting, perhaps, is the application of Bolzano's concept to an example made famous by Kant: $7 + 5 = 12$. This statement can be interpreted in a variety of ways: for example, if we think the base-10 representation important, it might be taken to mean the same as $7 + 5 = 10 + 2$. Or, if we abstract from this feature of the notation and rely simply on the definitions $2 = 1 + 1,\ 3 = 2 + 1,\ \ldots\ 12 = 11 + 1$, we could figure as follows (where df.= by definition, per. = by permutation; dis.= by disagreggation; ag. = by aggregation):[45]

[45] A similar proof may be found in *Bolzano 1810*, Appendix, §8. It should be noted, however, that he was then operating with a different notion of a whole. See *Bolzano 1977* for a presentation of his early theory.

$$\begin{array}{ll}
7+5 & \\
=7+(4+1) & df. \\
=7+4+1 & dis. \\
=7+1+4 & per. \\
=(7+1)+4 & ag. \\
=8+4 & df. \\
=8+(3+1) & df. \\
=8+3+1 & dis. \\
=8+1+3 & per. \\
=(8+1)+3 & ag.
\end{array}$$

$$\begin{array}{ll}
=9+3 & df. \\
=9+(2+1) & df. \\
=9+2+1 & dis. \\
=9+1+2 & per. \\
=(9+1)+2 & ag. \\
=10+2 & df. \\
=10+(1+1) & df. \\
=10+1+1 & dis. \\
=(10+1)+1 & ag. \\
=11+1 & df \\
=12 & df.
\end{array}$$

What is particularly interesting about Bolzano's way of looking at things is that according to him all of the above expressions not only refer to the same object (the same sum), but in addition *designate one and the same concept*. If he were inclined to speak as Kant did, he would say that the proposition $[7+5=12]$ was not only analytic but even identical. He says this:

> [T]he expression $A+B$ designates the same idea as $B+A$, and the expression $(A+B)+C$ the same idea as the following expressions: $A+(B+C), (A+C)+B, A+(C+B), (B+A)+C, B+(A+C), (B+C)+A, B+(C+A), (C+A)+B, C+(A+B), (C+B)+A, C+(B+A)$. For a sum is not altered when the parts of which it consists are connected in any arbitrary order.[46]

Just as, for example, the different expressions '$\{A, B\}$' and '$\{B, A\}$' might be taken to designate one and the same concept (the set containing exactly A and B, in no particular order), so too, Bolzano thinks, all the following expressions pick out the same concept:

$$12, 11+1, 10+2, 5+(5+2), (4+(3+2))+(2+1), etc.$$

He comments:

> It is certainly unfortunate that one and the same *idea* of a sum can be expressed in so many different ways that it is often only after lengthy consideration that one can gather that a single idea is designated by these expressions.[47]

[46] *Bolzano 1975*, §95 (p. 155).
[47] *Bolzano 1975*, §97 (p. 156).

If *statements* such as '7+5=12' are nevertheless informative, this must be at least in part because they teach us about the way certain signs are used. They may also help us to arrive at a more distinct understanding of the concept of a particular sum.[48]

Totalities In the *Theory of Quantity*, *totalities* are defined as a special kind of sum:

> A sum in which every object standing under the idea A, but no other object, appears as a part which is considered simple is called the *collection of all* A, of the *totality* of the A or the *entirety of* A in the concrete sense. [...] The universe gives us an example of a concrete totality of things, namely, the sum of all those substances whose actuality has a ground.[49]

A particularly interesting case is the totality of all objects, or the universal collection. Bolzano seems at one point to have thought that there could be no such collection. In one of his notebooks, we find the following:

> A remarkable example of a contradictory concept is that of the 'totality of things' or the sum of all things. In effect, this sum should belong to itself as a part, if it were indeed something. The same holds for the concept of the *totality* of *actual* things, because this totality would in turn have to be something actual.[50]

Later, however, he seems to have had second thoughts. In a marginal comment on the above paragraph, he writes:

> Not so! These concepts are not contradictory. Not at all! It follows from the concept of a collection that one may not take the same object twice, for example, the fingers on the hand and also the thumb, the index finger, the middle finger, the ring finger, and the pinky.[51]

As it stands, this retraction is elliptical. But perhaps we can guess what Bolzano thought he saw by reflecting further on the notion of a sum. The problem that needs

[48]Cf. *Bolzano 1837*, §447, no. 2 [IV:117]: 'We are ashamed when in daily life we find ourselves unwittingly making a merely identical judgment; this would be even more inexcusable for an author of a treatise, especially one who is setting out the essential theses of his science. This is by no means intended to say that in cases where the identity of concepts is concealed, as in many mathematical equations, we would not be justified in putting forth such a proposition (while expressly mentioning that it is identical). But in that case it is not really the proposition itself but rather the statement that it is an identical proposition that we teach on this occasion.'

[49]*Bolzano 1975*, §119 (p. 167). In *Bolzano 1837* (§86.3), we find a different definition of a totality, as the mere *collection* of all A

[50]*Bolzano 1979*, p. 29.

[51]Cited after *Sebestik 1992*, p. 316.

to be addressed, from one point of view, is that any attempt to collect all objects would necessarily fall short, because the alleged collection would itself have to be another object, which (according to the no-redundancy principle) could not be incorporated into itself as a part, because its parts would already belong to that collection.

If, however, we think of the collection of all things as a *sum*, there may be a way around this problem. In Bolzano's example, if we are dealing with a sum, then once we have the individual fingers, it would seem that we also have the collection of these fingers by aggregation. Similarly, it might be thought that once one has collected in a sum all objects apart from the collection of all objects, the latter collection is nonetheless present as one of the aspects of that sum, just as the collection of the fingers is.

Yet there is a difficulty here. For Bolzano seems always to have understood 'part' as an anti-reflexive relation, i.e., in such a way that there are no *improper* parts of collections (this would seem to follow from the no-redundancy principle in any case). If so, then we could not say that the sum of all things was a part of itself. Similarly in the case of the fingers: if these were the only relatively simple parts of a given sum, then we could not properly say that their collection was part of that sum. To the best of my knowledge, however, Bolzano never addressed this problem.

Lines considered wrt length The example of the line will be discussed in the next section. For the time being, I will simply reproduce his further remarks:

> [W]hen we only consider the length of the line, we look upon it as being composed of smaller lines whose manner of combination is a matter of indifference; and the parts of these smaller lines, insofar as they themselves are lines, can also be looked upon as parts of the original line.[52]

Odd sums The last two examples, finally, reflect Bolzano's customary, sometimes excessive, striving for maximal generality. They are meant to illustrate the breadth of his conception, according to which arbitrary objects may be combined to form a sum. He adds that, although such sums are admitted by his theory, it is difficult to conceive of circumstances in which it would be useful to consider them. This may be true for sums such as 3 cubic inches+5 days+2 hundredweight, but there are other odd sums that it might be worth thinking about. Take, for example, the sum of several machines, where the relatively simple parts are the smallest manufactured parts. Here, the sum would include, among many useless piles of random junk, also any machine that might be built by combining parts from the

[52] *Bolzano 1837*, §84 [I.400].

various machines (plus a remainder). Thus, for example, a working car might be assembled from the combined parts of two or three wrecks. The working car, along with the remaining parts, would then be one of the aspects of that particular sum.

Sums from the point of view of set theory

Although Bolzano's collections cannot be identified with modern sets, it is sometimes helpful, as Ali Behboud has remarked, to draw on the resources of set theory when interpreting Bolzano's theories.[53] I shall do that here in an attempt to clarify Bolzano's definition somewhat.

To begin with, I will describe sets that may go proxy for Bolzanian sums composed of a finite number of (relatively) simple parts. I assume that we are working in set-theory with ur-elements. Consider a set S whose only elements are ur-elements and which has 2 or more elements. The idea is that these will be the ultimate parts of a sum, those which, as Bolzano puts it, are regarded as simple. Call such a set S a *sum-domain*. For the sake of convenience, I also define a *B-proper subset* as a proper subset with at least two elements.

Now consider the following three operations on sets:

1. PERMUTATION of elements: e.g., from $\{A, B, C\}$ to $\{B, C, A\}$. (Invariance under permutation is of course built into the concept of a set.)

2. DISAGGREGATION of elements. Replacement of an element with elements by those elements. Applying this rule to the set $\{A, B, \{C, D\}\}$, for example, we obtain $\{A, B, C, D\}$. (This operations interprets Bolzano's 'the parts of a given part may be considered parts of the whole'.)

3. AGGREGATION of elements. Replacement of any B-proper subset of elements of a given set by the set containing these elements. For example, if $S = \{A, B, C\}$ and $T = \{B, C\}$, then this operation yields $\{A, \{B, C\}\}$. (This is the inverse operation of (2), which Bolzano in my opinion tacitly assumes to be permissible with sums.)

Now define a relation ρ as follows. $X \rho Y$ iff Y may be obtained from X by finitely many applications of the operations 1-3. This relation is clearly reflexive, symmetrical, and transitive. Given a finite sum-domain S, the associated equivalence class under ρ, $[S]_\rho$ can then serve as a set-theoretical Ersatz for the sum whose ultimate parts are the elements of S.

[53] *Behboud 1997*, p. 114–115.

The simplest case is a sum with two (relatively) simple elements, A and B. Since there are no B-proper subsets here and only ur-elements, operations 2 and 3 remain idle. The set-theoretical Ersatz will be the singleton

$$\{\{A, B\}\}$$

Though it might be disputed whether this is in fact a sum, since it would only satisfy the sum-principle vacuously.

With 3 ur-elements, we have the following equivalence class:

$$\{\{A, B, C\}, \{A, \{B, C\}\}, \{B, \{A, C\}\}, \{C, \{A, B\}\}\}$$

corresponding to the following array of expressions:

$$\begin{array}{llll} A+B+C & A+(B+C) & B+(A+C) & C+(A+B) \\ A+C+B & A+(C+B) & B+(C+A) & C+(B+A) \\ B+A+C & (B+C)+A & (A+C)+B & (A+B)+C \\ B+C+A & (C+B)+A & (C+A)+B & (B+A)+C \\ C+A+B & & & \\ C+B+A & & & \end{array}$$

With four, the already quite messy:

$$\begin{array}{l} \{\{A, B, C, D\}, \{A, \{B, C, D\}\}, \{B, \{A, C, D\}\}, \{C, \{A, B, D\}\}, \\ \{D, \{A, B, C\}\}, \{\{A, B\}, C, D\}, \{\{A, C\}, B, D\}, \{\{A, D\}, B, C\}, \\ \{\{B, C\}, A, D\}, \{\{B, D\}, A, C\}, \{\{C, D\}, A, B\}, \{\{A, B\}, \{C, D\}\}, \\ \{\{A, C\}, \{B, D\}\}, \{\{A, D\}, \{B, C\}\}, \{A, \{B, \{C, D\}\}\}, \\ \{A, \{C, \{B, D\}\}\}, \{A, \{D, \{B, C\}\}\}, \{B, \{A, \{C, D\}\}\}, \\ \{B, \{C, \{A, D\}\}\}, \{B, \{D, \{A, C\}\}\}, \{C, \{A, \{B, D\}\}\}, \\ \{C, \{B, \{A, D\}\}\}, \{C, \{D, \{A, B\}\}\}, \{D, \{A, \{B, C\}\}\}, \\ \{D, \{B, \{A, C\}\}\}, \{D, \{C, \{A, B\}\}\}\} \end{array}$$

And so on.

And one might now attempt to extend the definitions given above by permitting infinite sum-domains and redefining the relation ρ by permitting transfinite sequences of the three operations.

Even so, it seems to me that we would not have taken the full measure of Bolzano's conception of a sum. For, as was his wont, Bolzano aimed at even greater generality, as is indicated by the example of a line considered with respect to its length. In this case, we have a whole (a line segment) which may be partitioned however we like into smaller line segments. Since a division into points is ruled out by the concept of this particular sum and each one of these parts is also further divisible into other homogeneous parts, we see in this case a sum with a completely different structure, lacking ultimate, (relatively or absolutely) simple parts.

Yet the same possibilities of rearrangement, dissolution and fusion are also apparent here. Consider a simple partition of the line $\overline{AB}$ into two halves, $\overline{AC}$ and $\overline{CB}$. (To avoid the problem of segments sharing endpoints, I will adopt the convention that the left endpoint belongs to a segment, but the right one does not; that is, I will identify the segment $\overline{AB}$ with the half-open interval of points $[A, B)$.) Thus we have a partition of $[A, B)$ into $[A, C) + [C, B)$.

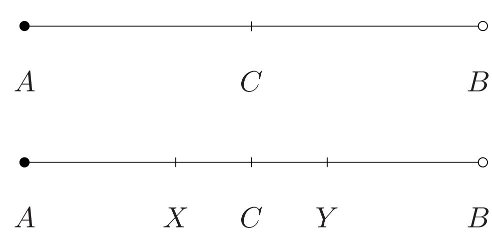
Next, consider a completely distinct partition of the line, say into thirds:

$$[A, B) = [A, X) + [X, Y) + [Y, B)$$

Suppose now, as seems reasonable, that $[A, C)$ can be subdivided into $[A, X) + [X, C)$ and $[C, B)$ into $[C, Y) + [Y, B)$. Thus we have:

$$[A, B) = [A, X) + [X, C) + [C, Y) + [Y, B)$$

If we now fuse the two middle parts, i.e., $[X, C) + [C, Y)$, we obtain $[X, Y)$, and thus have a partition into thirds.



Proceeding in this way, we can obtain any partition of the line we wish into smaller line segments, even one with infinitely many parts, as would be obtained, for example, by repeatedly halving the rightmost interval. It would also be permissible to rearrange parts, e.g., exchanging the places of $[A, X)$ and $[Y, B)$ or, indeed, by reconfiguring them however one likes. And, just as desired, the total length remains invariant under any of these transformations.

The way in which Bolzano specifies such sums is noteworthy. In the examples considered above, a sum was determined by indicating some partition or other, along with its sub-partitions, their sub-partitions, and so on. This could also be done in the present case: we might begin, for example, by dividing the interval $[A, B)$ into two halves, then proceed by dividing these halves into halves, and so on *ad infinitum*. If we then allowed infinite sums, we could obtain whichever other partition we wished via the operations of disaggregation, permutation, and aggregation. Interestingly, however, Bolzano does not proceed in this way. Instead, he simply notes that any (exhaustive) division of a line into smaller lines is to count as a partition. Thus he seems at least implicitly to appeal to the absolute notion of a sum-part mentioned above (section 4), namely that of a part relative to some partition or other.

Conclusion

If my interpretation is correct, sums, just like multitudes in general, are collections that retain their identity under certain kinds of transformations. Crucially, these transformations include aggregation or fusion as well as disaggregation or dissolution. As mentioned above, I think Bolzano most likely believed that invariance under aggregation followed from the concept of a sum, defined as a multitude with the additional feature that the parts of its parts may be considered parts of the whole. I also think he assumed that this would be obvious to his readers.

In my opinion, he was wrong on both counts. Regarding the first point, consider a sum with three relatively simple parts, represented as $A+B+C$. Bolzano appears to think that we can transform $A + B + C$ into $A + (B + C)$ without affecting the identity of the sum. But what, if not invariance under aggregation, justifies this operation? Not the principle that parts of parts may be considered parts of the whole, for we cannot say that $B + C$ is a part of a part of $A + B + C$, unless we are prepared to say that $A + B + C$ is an (improper) part of itself—but this would violate the no-redundancy principle. With respect to the second point, the fact that a number of extremely intelligent readers who know Bolzano well did not find in his remarks any hint of the principle of invariance under aggregation speaks for itself.

Bolzano's exposition, then, was not all that it should have been. His concept of a sum, however, remains quite interesting and worthy of further investigation. To begin with, it would be instructive to reconsider his presentation of the foundations of arithmetic in the light of this interpretation. With the example of the line in mind, it might also be fruitful to consider whether, with a suitable modification, Bolzano's notion of a sum might be put to use in the theory of measure and integration. But these are separate inquiries, best left for another occasion.

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