A Last Shot at Kant and Incongruent Counterparts

Paul Rusnock and Rolf George

In *Kant-studien* **86** (1995) 257–277.

§ 1

Kant's deliberations on incongruent counterparts, left and right hands and the like, seem to us to point to a problem so well circumscribed that a decisive treatment of it should be possible, despite the division of opinion one finds in the voluminous literature on the subject.[1] Felix Mühlhölzer's "Das Phänomen der inkongruenten Gegenstücke aus Kantischer und heutiger Sicht"[2] goes a long way toward a satisfactory account of Kant's several arguments, and gives an interesting discussion of more recent developments in mathematics that yield an elegant solution of the problem which occasioned Kant's puzzlement. More controversial is his suggestion that Kant, lacking modern concepts, could not have produced a more adequate answer than he did.

Recently a number of commentators have attempted to reformulate Kant's remarks in the light of later developments, for instance by reference to models of space other than three-dimensional Euclidean space, including higher dimensional or non-orientable spaces, or to parity violating reactions in particle physics.[3] But there is no reason to suppose that Kant's arguments involve tacit appeals to such exotica. There is, rather, every indication that Kant thought of space much as Euclid conceived it: i.e. 3-Dimensional, infinite, orientable, homogenous. While reference to other models of spatial relations may help commentators clarify a few related issues, no such thought should be attributed to Kant himself.

Our aim is more narrowly exegetical. We want to explicate Kant's terms and arguments, give a plausible account of their sources, and discuss their historical context. We expect to show that Kant's problem was readily solvable with the conceptual resources available to him. We will give some indication of why Kant failed to recognize this and, furthermore, why he was not notably disturbed by this failure.

We shall be concerned in the main with the second of Kant's six discussions of, or allusions to, incongruent counterparts, the 1768 essay "Von dem ersten Grunde

[1] For instance the essays recently collected in J. Van Cleve and R.E. Frederick eds., *The Philosophy of Right and Left: Incongruent Counterparts and the Nature of Space* (Dordrecht: Kluwer, 1991), and the bibliography in that volume.

[2] *Kant-Studien* 83(1992)436-453.

[3] This is true of several of the essays in Van Cleve and Frederick and also of J. Vance Buroker, *Space and Incongruence: The Origin of Kant's Idealism* (Dordrecht: Reidel, 1981), 82 f.

des Unterschiedes der Gegenden im Raume" (2.377-383).[4] We shall refer to it as "Directions", preferring this translation of "*Gegend*" to the customary "region" for reasons given below (§7).

In "Directions" Kant defines incongruent counterparts thus: "When a body is perfectly equal and similar to another, and yet cannot be included within the same boundaries, I entitle it the incongruent counterpart of that other."[5] The following examples are furnished: a hand and its mirror image, an ear and its mirror image, certain pairs of spherical triangles. Kant then claims that there is an "inner" distinction between such pairs, because they are incongruent: "Since this surface bounds the physical space of the one but cannot serve as boundary to the other, however one may turn and twist it, this difference must be such as rests upon an inner ground."[6] Kant claims that incongruent counterparts are *equal* and *similar* but *incongruent*; that they are identical as regards some, but evidently not all, their *inner* characteristics. In order to understand his argument, we will need first to clarify his use of these terms. They have, as one might expect, their roots in antiquity.

§**2**

In Euclid's *Elements*, the concept of congruence does not appear, although the notion of coincidence does.[7] One can reconstruct the concept of congruence from the practice of Greek mathematics in the following way: figures which can be "applied" to one another to coincide are congruent. The notion of application evidently involves a principle of motion, and it is clear from, e.g. *Elements* I, 4, that the motions in question are rigid motions in 3-space.[8] "Equal", especially when used in conjunction with "similar", means "equal in magnitude". Archimedes, for

[4]References of the form (volume.page) are to the Academy edition of Kant's works. Kant's other discussions are in the Metaphysik Herder of 1763, 28.15, the *Inaugural dissertation* §15C (2.402-403), *Prolegomena* §13 (4.285-286), *Metaphysische Anfangsgründe der Naturwissenschaft* , (4.483-4) and *Was heißt: sich im Denken orientieren?* (8.134-135), hereafter "Orientation". We follow these translations: (1) "*Directions*" and *Dissertation*: J. Handyside, *Kant's Inaugural Dissertation and Early Writings on Space* (Chicago: Open Court, 1929); (2) *Prolegomena*: Carus (revised), *Prolegomena to any Future Metaphysics* (Indianapolis: Bobbs, Merill, 1949); (3) "*Orientation*": L.W. Beck, *Critique of Practical Reason and other Writings in Moral Philosophy* (Chicago: University of Chicago Press, 1950); (4) *Anfangsgründe*: J. Ellington, *Metaphysical Foundations of Natural Science* (Indianapolis: Bobbs-Merrill, 1970).

[5]2.382; Handyside, 26.

[6]2.382; Handyside, 27.

[7]E.g. in Common Notion 4: "Things which coincide with one another are equal." T.L. Heath ed. and trans., *The Thirteen Books of Euclid's Elements* (New York: Dover, 1956), I, 155; for other examples of the use of coincidence in antiquity, cf. *Elements*, Book I, props. 4 and 8; also, Archimedes, "Of conoids and spheroids," Prop. XVIII; "On the equilibrium of planes," I, postulate IV.

[8]Cf. Heath, I, 224f.

instance, has: "Any circle is equal to the right triangle with one leg equal to its radius and the other equal to its circumference."[9]

More importantly, the term "similar", although used in many contexts, is defined only for a handful of figures. Euclid gives separate definitions for similar segments of circles, rectilineal figures, plane and solid numbers, solid figures, cones and cylinders.[10] The use of the term "*homoios*" in geometrical contexts suggests something like "sameness of form", apparently a general concept (since the same word is always used): *but there is no general definition.*

There seems to be an assumption underlying the several Euclidean definitions of similarity, namely, that spatial figures which are both similar and equal are congruent. Put another way, similar figures should differ at most in size (or scale). The existence of incongruent counterparts therefore points to a defect in Euclid's definition of equal and similar solids (i.e. polyhedra): "**Equal and similar solid figures** are those contained by similar planes equal in multitude and in magnitude:"[11] there are polyhedra which by this definition are equal and similar, but incongruent.

This defect may already have been noticed in antiquity. Heron of Alexandria adds to Euclid's definition the stipulation that the bounding planes be *similarly situated*.[12] A related problem in defining similarity for 3-dimensional objects also occurs in the study of spherical triangles, which can, as Kant remarked, have equal angles and sides but fail to be congruent. Menelaus of Alexandria (ca. 100 A.D.), who wrote a treatise on *Spherics* (which survives only in an Arabic translation) may have noticed this. Cantor thinks he probably did; Heath disagrees.[13] Among Kant's contemporaries, Segner in 1741 and Karsten in 1760 had published this

[9] Archimedes, "On the Measurement of the circle", Prop. 1.

[10] "**Similar segments of circles** are those which admit equal angles, or in which the angles are equal to one another." (Book III, defn 11., Heath, II, 2); "**Similar rectilineal figures** are such as have their angles severally equal and the sides about the equal angles proportional."(VI, defn 1; Heath, II, 188); "**Similar plane** and **solid** numbers are those which have their sides proportional."(Book VII ,defn 21; Heath, II, 278); "**Similar cones and cylinders** are those in which the axes and the diameters of the bases are proportional." (Book XI, Definition 24, Heath, III, 262).

[11] Book XI ,defn. 10; Heath, III, 261.

[12] *Heronis Alexandrini: Opera quae Supersunt Omnia,* Vol IV, *Heronis Definitiones* ed. J.L. Heiberg (Reprinted Stuttgart: Teubner, 1976), 75-6. Stimson, in his 1756 edition of the *Elements*, found the definition defective on other grounds; cf. Heath, III, 265. Kant's problem reduces to one of characterizing differences in point sets. Euclid's problem (i.e. that of giving necessary and sufficient conditions for the congruence of polyhedra), which is much harder and considerably more interesting, was first seriously taken up by Legendre and Cauchy. For a discussion of Euclid's definition and its subsequent history, see Sigfried Heller, "†ber Euklids Definitionen ähnlicher und kongruenter Polyeder," *Janus* 5(1964) 277-90.

[13] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, 3rd ed., (Reprint: New York: Johnson Reprint Co., 1965), I, 413; T.L. Heath, *History of Greek Mathematics* (Oxford, 1921; reprinted New York: Dover, 1981), II, 263.

observation.[14]

In addition to these difficulties, the Euclidean definitions suffer from the significant defect of being only piecemeal; no provision is made for parabolas, spirals, etc. They do not determine when two points, two lines, two spheres are similar; in every case one must formulate a new definition. The individual definitions of similarity, pointing as they do to a general notion, have more the character of theorems (requiring proof!) than of definitions. That is, given that the conjunction of similarity and equality is supposed to yield congruence, the definitions can be seen as theorems about the conditions, in addition to equality, various figures must satisfy for congruence. In order to develop such theorems properly, however, one should first have an adequate mathematical definition of similarity.

Leibniz, reflecting on the use of these notions in Greek geometry, was perhaps the first to provide such a general definition. In a series of notes aimed at developing a mathematics of situation (*analysis situs*[15]) and a corresponding symbolic calculus (*characteristica geometrica*), he suggests a general division of geometrical concepts into those pertaining to form (quality) and those pertaining to quantity. Judgments of quantity in geometry, he notes, always involve comparisons—that is, one says that the areas of two circles stand in the ratio of 2 to 1, but not that either circle alone has some particular area. Common speech may suggest otherwise (e.g. when it is said that a tree is so many metres high), but such statements involve units, which are simply standard comparison objects. Considered in isolation, then, no geometrical figure has any particular size; yet many things may still be said of it. For instance, if it is a polygon, the number of its sides, the size of its angles, and the ratios between any two homogeneous magnitudes determined by the figure (e.g., that between two diagonals) can all be determined without reference to other figures. All of these things, which can be discerned in the figure (and in the figures determined by the given figure) without comparing it to other figures, according to Leibniz, may justly be said to belong to its form.

Given these general observations, Leibniz then defines similarity as equality of form, i.e. two geometrical figures are *similar* if and only if, considered alone, they are indistinguishable.

[14] According to H. Vaihinger, *Commentar zu Kants Kritik der reinen Vernunft* (Stuttgart, 1881/1892, reprinted New York: Garland, 1976), II, 531n, where Segner is also identified as a possible source of Kant's acquaintance with counterparts.

[15] Leibniz's works are cited from the following editions: *Phil.=Die philosophische Schriften von G.W. Leibniz* , ed. Gerhardt (Berlin, 1875-90); *Math. = Leibnizens mathematische Schriften* , ed. Gerhardt (Berlin-Halle, 1849-1863); Loemker = *G.W. Leibniz: Philosophical Papers and Letters*, L. Loemker ed. and trans. (Chicago, University of Chicago Press, 1956). For examples of Leibniz's *analysis situs* see *Phil.*, V, 178 f.; Loemker, I, 392; for more detailed references, see L. Couturat, *La Logique de Leibniz* (Paris, 1901; Reprint Hildesheim: Olms, 1961), 310f, 388f ; also, H. Freudenthal, "Leibniz und die analysis situs," *Studia Leibnitiana* 4, (1972), 61-69.

> In undertaking an explanation of quality or form, I have learned that the matter reduces to this: things are *similar* which cannot be distinguished when observed in isolation from each other. Quantity can be grasped only when the things are actually present together or when some intervening thing can be applied to both. But quality presents something to the mind which can be known in a thing separately and can then be applied to the comparison of two things without actually bringing the two together either immediately or through the mediation of a third object as a measure.[16]

The characteristics which can be discerned in figures considered in isolation are *inner* characteristics; all the rest, that is, those which may be determined only via comparison, are *external* characteristics.[17] It is sometimes suggested that inner characteristics are non-relational (presumably expressible as monadic predicates) or those which involve only the points of a given figure.[18] But this is a mistake. To say that something is an inner characteristic entails neither of these things. For instance, that the tangents drawn to a circle on opposite ends of a diameter are parallel is an *inner* characteristic of the circle, since the tangents are determined by the circle. Inner characteristics are non-relational only in the sense that they do not depend on a comparison with figures which are not determined by the given figure. In Kant's 1768 essay, these characteristics are said to pertain to *position*.[19]

Figures which have the same inner characteristics are called *similar*. Figures are *congruent* when capable of being moved to coincide, or when they differ at most by being in a different place (*solo numero*).[20] *Equality* is still simply equality of magnitude.[21] Leibniz believed congruence to be definable as the conjunction of similarity and equality.[22]

[16] *Phil.* V, 179; Loemker, I, 392.

[17] Cf. his remarks on intrinsic and extrinsic denominations, e.g. letter to De Volder, 20 June, 1703, *Phil.* II, 248 f.; Loemker, II, 861. Leibniz's methodological principle is a promising one for Euclidean geometry; for, as Wallis had shown, the independence of form and quantity (i.e. the existence of unequal similar figures) is equivalent to the parallel postulate. J. Wallis, "Demonstratio postulati quinti Euclidis" (1663), in *Opera Mathematica* (Oxford, 1693; reprint Hildesheim: Olms, 1972), II, 674 ff.

[18] For instances, see the following in Van Cleve and Frederick: Nerlich, p.157f., esp. 164-5; Earman, 138-9; Sklar, 174 f.; Walker, 190 f.; Van Cleve, 204-5; also, Mühlhölzer, 452. The argument (found in Nerlich, 158; Walker, 191; Mühlhölzer, 452) that orientation is not an inner characteristic because, for example, a hand and its mirror image are congruent in Euclidean 4-space but not in Euclidean 3-space, depends upon the mistaken assumption that inner characteristics make no reference to points outside the given figure.

[19] 2.377-8.

[20] *Math.*, V, 265; also, *Math.*, VII, 29, 263; cf. Couturat, 310-311.

[21] Cf. Couturat, 312 for amplification.

[22] E.g. *Math.*, V, 154, 179, 275; cf. Couturat, 311n4.

From his writings it is clear that Leibniz thought that the inner characteristics were exhausted by number (e.g. of sides of a polygon), angle, proportion, and the external characteristics by (relative) magnitude.[23] As Kant's examples show, however, this enumeration involves an oversight, for there are pairs of figures which are similar and equal under these definitions and yet fail to be congruent. Like Euclid and many others, Leibniz did not notice that generalization from the case of the plane (thought of as embedded in 3-space) to 3-space raises a new question.

Leibniz sent some brief notes on *analysis situs* to his mathematical correspondents, but his work became known, for the most part, through the mediation of Christian Wolff,[24]who introduced Leibniz's definitions into his systems of mathematics and metaphysics, and thus to the wider German philosophical community.[25] In his *Elementa Matheseos Universae,* Wolff defines *Similars* as those things which are identical in those characteristics which can be discerned in things considered by themselves.[26] In the German version of his mathematics, Wolff expresses this somewhat differently: "Similarity is the agreement of that by means of which things are differentiated *by the intellect*."[27] He notes that, since quantity is not determinable without reference to other things, similars may differ in quantity, and concludes that they can *only* differ in quantity;[28] and, finally, that *this* difference cannot be grasped by the intellect: "One can give someone a magnitude, and indis-

[23]If figures are conceived as point sets or loci, we can formulate Leibniz's definition of similarity as follows. Two spatial objects are similar if and only if, given four points in (or determined by) one $\langle p_1, p_2, p_3, p_4 \rangle$ and the four corresponding points in (or determined in the same way by) the other $\langle p'_1, p'_2, p'_3, p'_4 \rangle$ the corresponding distances are in the same ratio (i.e., $d(p_1, p_2) : d(p_3, p_4) :: d(p'_1, p'_2) : d(p'_3, p'_4)$, etc.), and the angles determined by triplets of corresponding points are equal (i.e., $\angle p_1 p_2 p_3 = \angle p'_1 p'_2 p'_3$, etc.)

[24]A number of people, including Kant (2.377) had heard the name of Leibniz's proposed science, but had seen nothing of it. For the chaotic subsequent history of the term *analysis situs*, cf. Freudenthal, *Op. cit.*

[25]Wolff's views on Similarity were published in the *Acta Eruditorum*, A, 1715, also in his *Elementa Matheseos Universae* (Halle, 1742; Reprinted Hildesheim: G. Olms, 1968), see especially *Arithm,* §§24-26; *Geom.* §§3,119,120,161,162,564; *Philosophia Prima Sive Ontologia (*Frankfurt and Leipzig, 1736; Reprinted Hildesheim: Olms, 1977), Sect. II, Cap. 1, "*De identitate & Similitudine*"; cf His *Mathematisches Lexicon* (Leipzig, 1716; reprint Hildesheim: Olms, 1965), 1278f.

[26]*Elementae Matheseos Universae, Elementa Arithmeticae* Cap. 1, §24; cf. *Ontologia* §195 f.; Also, *Mathematisches Lexicon*, 1278; finally, *Vernünftigen Gedancken von Gott, der Welt und der Seele des Menschen, auch allen Dingen überhaupt*, (Halle, 1751; Reprint Hildesheim: Olms, 1983), §18.

[27]*Anfangsgründe aller mathematischen Wissenschaften* I. 7th ed. (Frankfurt and Leipzig, 1750; reprinted Hildesheim: Olms, 1973), I. Abt., Bd. 12, 118.

[28]*Elementae Matheseos Universae, Elem. Arith.* §26. Cor. 2: Cum quantitas sine alio assumto per se non intelligi, sed tantum dari possit (§13, 14); Similia, salva similitudine, quantitate differre possunt (§25), atque adeo quantitas est discrimen internum similium; *Elementa Geometriae*, Cap. II, §162: "Similia differe nequeunt, nisi quantitate (§26, Arith.)."

tinctly grasp it in the imagination, but one cannot explain it in words, and clearly grasp it with the understanding."[29] His emphasis on intellect and concepts is a likely source for Kant's remarks in his later writings that the difference between incongruent counterparts is not expressible conceptually and thus not accessible to the understanding. But Wolff, too, failed to notice the existence of incongruent counterparts.

§.3.

Kant noticed the problem that had escaped Euclid, Leibniz, and Wolff around 1762/63. In Herder's transcripts of his metaphysics lectures we find the following comment on Baumgarten's *Metaphysica*, §70:

> Congruentia: figures that are equal and similar to each other. . . .
>
> The concepts of congruence are extended [erweitert] in mathematics. [Objects] equal and similar are not congruent unless they lie *in a plane* [Aequalia et similia congruunt non nisi *in plano*.] (28.15).

We shall return to this in a moment. At the time of the lecture Kant was writing, or had already written, his Prize Essay for the Berlin Academy "On the Clarity of the Principles of Natural Theology and Morals."[30] The Academy had asked whether the truths of metaphysics are capable "of the same clear proof as geometric truths" (2.493), thus inviting a comparison of the philosophical and mathematical method. Kant's response was that the method of philosophy is "analytic," that of mathematics "synthetic." He said this:

> Mathematicians have. . . sometimes given analytic explanations, I admit; but it has always been an error as well. It was in this way that Wolff considered similarity in geometry with the eye of a philosopher, intending to subsume the geometrical concept of similarity under the general notion of similarity. He could, however, have saved himself the trouble; for when I think of geometrical figures in which the angles included by the lines of the outline are reciprocally equal, and the sides which they include [are in the same ratio], then this can always be regarded as the definition of the similarity of these geometrical figures; and the same holds for the remaining similarities of spaces. The

[29] *Mathematisches Lexicon*, 1179.

[30] Herder attended his first lecture on 21 August, 1762, when Kant was finishing his metaphysics course with reflections on witches, kobolds, and poltergeists (§796 of Baumgarten's *Metaphysica*, 28.148). Incongruence is treated in §70 of Baumgarten, and may have come up later in 1762, in the next cycle of lectures. At that time Kant was writing the Prize Essay.

> general definition of similarity is of no consequence whatever to the geometer. It is fortunate for mathematics that, even though the geometer engages from time to time in such analytical explanations as a result of a false idea of his task, in the end nothing is actually deduced by him, or, if so, his most immediate inferences constitute, basically, the mathematical definition. Otherwise this branch of knowledge would be liable to the same unfortunate discord as philosophy itself. [31]

Kant here not only criticizes Wolff (and Leibniz), but in fact endorses the Euclidean piecemeal procedure as the very essence of mathematics. That there is a general concept or family of concepts of similarity in geometry is not doubted. The question is how one can get a grasp of it. As Kant sees it, the geometer generates this concept or family of concepts by a cumulative, ideally exhaustive, account of "similarity" for different types of figures and bodies. If a non-cumulative account of similarity (or of some other mathematical notion) could be developed, it would have to be by *analysis*, that is, as Kant will later say, by "becoming aware of the components which [one] always thinks in [the concept]".[32]

The alternative to this procedure, in 1762, is not, as later, an appeal to intuition, but the cumulative, or synthetic, method. This is, in fact, his first attempt at resolving the problem of incongruent counterparts: if there is no general *concept* of similarity, then there is also no reason to expect a general *theorem* to the effect that similar and equal figures are congruent. (Note his consistent use of the plural: "*concepts* of similarity" in the lecture, "*similarities*" in the Essay). These concepts are generated by "extending" a concept already in hand to other cases: the mathematician begins with plane figures, typically triangles, and discovers that similar and equal ones are congruent. He then investigates polygons, solids, etc., generating *related but different* concepts. Geometrical similarity is thus not a concept, but a family of concepts that resemble each other in some ways, differ in others, and do not admit certain general theorems, specifically not the core theorem of *analysis situs*, that similar and equal figures are congruent.

The Herder lecture suggests that Kant's account of mathematical method in the Prize Essay was motivated by the discovery of incongruent counterparts. Unfortunately, his clever Wittgensteinian solution—no general theorem where there is only family resemblance—rests on a confusion which Bolzano, for example, never tired of reproving and which, in Coffa's opinion, it took most of the 19th century

[31] 2.277; Quoted after D.E. Walford tr., "Enquiry concerning the clarity of the principles of natural theology and ethics," in G.B. Kerford and D.E. Walford eds., *Kant: Selected Pre-Critical Writings and Correspondence with Beck* (Manchester: Manchester University Press, 1968), 7.

[32] *Critique of Pure Reason*, A7/B11.

to recognize and neutralize.[33] It relies, namely, on the identification of conceptual truth with analytic truth in the narrow Kantian sense. If anything is based on concepts alone, it is the modern combinatorial resolution of Kant's incongruence problem. But this does not mean that that resolution is based on the *analysis*, that is decomposition, of concepts such as similarity, already in Kant's possession.

§.4.

This explanation could not long satisfy Kant for, even if it were correct, it does not address the problem of accounting for the incongruence of similar and equal solids. "Directions" of 1768 considers this problem and sketches a solution. Reverting without apology to Wolff's definition, and without further reference to the cumulative procedure, he writes:

> The right hand is similar and equal to the left, and *if we look at one of them alone by itself*, at the proportions and positions of its parts relatively to one another and at the magnitude of the whole, a complete description of it must also hold for the other in every respect.[34]

Now incongruent counterparts, such as the perfectly symmetrical hands contemplated here, are similar according to the definition given above; they are also equal. They should therefore be congruent, which they obviously are not. We now have a counterexample—the 1764 essay had cited nothing specific, though he must surely have had cases like these in mind—which shows the centrepiece of Leibnizian-Wolffian geometry to be an antitheorem.

Two obvious options present themselves at this point: 1) any or all of the definitions of "similar", "equal", "congruent" could be changed so that the central theorem still holds; or 2) the central theorem could be reformulated. In the first case, since "equality" and "congruence" seem unproblematic, the most promising move would be to include orientation (or "handedness", "the direction towards which the parts are ordered"[35]) of figures as a property belonging under the general heading of similarity. In received terminology, this would mean adding orientation to the list of *inner* characteristics of figures. In the second case, the congruence theorem would add, as Heron did, an additional condition: "figures are congruent if and only if they are similar and equal and of the same orientation." In this

[33] J. Alberto Coffa, *The Semantic Tradition From Kant to Carnap: to the Vienna Station* (Cambridge: Cambridge University Press, 1991), 21.

[34] 2.381; Handyside, 26; cf. *Prolegomena*, 4.285-6; Beck, 33: "For instance, two spherical triangles on opposite hemispheres which have an arc of the equator as their common base may be quite equal, both as regards sides and angles, so that nothing is found in either, if it be described for itself alone and completely, that would not equally be applicable to both. . . ."

[35] ". . . die Gegend nach welcher sie geordnet sind. . . ," 2.377.

case, "having the same/different orientation" is added as an *outer* characteristic. It seems reasonable to suppose that Euclid would have adopted the former, Leibniz the latter, repair to their systems.

Kant, however, seems to have wanted it both ways. His further problems, and the puzzlement of generations of subsequent commentators, can be traced to his inability to make up his mind. On the one hand, he consistently claims that the difference between counterparts must rest on an inner ground:

> . . . there remains an inner difference, namely, that the surface which bounds the one cannot possibly bound the other. Since this surface bounds the physical space of the one but cannot serve as boundary to the other, however one may turn and twist it, this difference must be such as rests upon an inner ground.[36]

On the other, he writes that this characteristic is one which "can be apprehended only through comparison with other bodies,"[37] which is precisely Leibniz's and Wolff's definition of an outer characteristic. The same tension may be observed in the thought experiment of the lone hand Kant offers to clinch his argument in the 1768 essay:

> . . . if we conceive the first created thing to be a human hand, it is necessarily either a right or a left, and to produce the one a different act of the creating cause is required from that whereby its counterpart can come into being.[38]

This suggests that handedness is an inner characteristic, since a lone hand would have it in isolation. But how do we know this? Again, it is by comparison: were the hand neither right nor left, he suggests, it would fit either side of a human body.[39] Orientation is thus a curious thing according to Kant: it is apprehended as if it were an outer characteristic (through comparison), but it is really inner.

We can now see where the difficulty lies. Kant says repeatedly that the difference between incongruent counterparts must rest upon an inner ground. This

[36] 2.382; Handyside, 27; cf. 4.286.

[37] 2.383; Handyside, 28.

[38] 2.382; Handyside, 27.

[39] P. Remnant, "Incongruent Counterparts and Absolute Space," *Mind* 72(1963), 393-399 (reprinted in Van Cleve and Frederick, 51 f.), 57, correctly notes that Kant's stated argument concerning the lone hand involves a *petitio principii*: "We can now see where Kant's own argument goes wrong: it involves the inconsistency of maintaining that it is impossible to say of a hand, considered entirely in isolation from everything else, whether it is right or left, while assuming that it would be possible to say of a handless body, considered by itself, which was its right side and which its left." Cf. Mühlhölzer, 443.

is so, he claims, because they are equal but incongruent. Now two figures are incongruent if they cannot be moved to coincide. But the incongruence of equal figures will entail an inner difference only if there are no outer differences other than magnitude and location. *Kant's argument, in other words, assumes that external characteristics are exhausted by magnitude and place.*

What of inner characteristics, then? When he says that the difference between counterparts is based on an inner ground *because* no twisting and turning can make them coincide, he evidently takes "internal" simply to be the negation of "external", i.e. all characteristics of spatial figures which are not outer are inner, and conversely. In this sense, inner characteristics are those that arise neither from magnitude nor from position. But he also uses Wolff's definition of inner characteristics as those which may be determined by consideration of the relations of the parts of a figure without comparing it to other figures.

The equivocation on the terms "inner" and "outer" is now apparent: "Inner difference" is used in the sense of "determinable by consideration of the relations of the parts of a figure considered alone" and also in the sense of "neither inequality nor difference of place." "Outer difference" is used in the sense of "not determinable by consideration of the relations of the parts of a figure considered alone," and also in the sense "either difference of magnitude or of place." The proposition: "spatial objects are congruent if and only if they agree in all inner and outer characteristics" is tautologous on both the first and second uses of the terms. If, however, the first and second uses are mixed, and "inner" is taken in the first sense ("determinable by consideration of the parts of a figure considered alone") and "outer" in the second ("either magnitude or difference of place"), then the proposition is false, as incongruent counterparts show. The "paradox"[40] of incongruent counterparts is thus a mere paralogism.[41]

§. 5.

With this clarification, it is an easy matter to dispose of the question of the "lone hand." If it is decided that handedness is an inner characteristic, then the lone

[40] 4.285.

[41] In more detail, let $inner_1$ =determinable by comparison of parts etc.
$outer_1$ = all characteristics which are not $inner_1$
$inner_2$ = all characteristics which are not $outer_2$
$outer_2$ = relative magnitude or difference of place

Kant's "Paradox" is as follows:

Figures which are identical in all $inner_1$ and $outer_1$ characteristics are congruent.
Counterparts are identical in all $inner_1$ and $outer_2$ characteristics.
They are therefore congruent.
But they are incongruent.

Therefore, etc.

hand will be either right or left, if an outer, then only the difference or sameness of handedness will be discernible. In the latter case, the relation "in the same sense" can be used to define two equivalence classes (say, on ordered triples of linearly independent vectors, or tripods)[42], one of which may be called (*ad libitum*) "right", and the other "left". If there is only one figure, a solitary hand (conceived as a tripod), it will still be possible to define the equivalence classes, but one of them will have only one member and the other will be empty. Since the names "right" and "left" can be assigned as one likes, the hand is, in a manner of speaking, neither "right" nor "left".[43] The question, "is this a right or a left hand" has a non-trivial meaning in this case only when one already has equivalence classes defined on part of a domain and seeks to extend them. This view of things seems consonant with both experience and Kant's expressed views, i.e. most people carry about exemplars of objects of different sense (i.e. their hands), and can determine the handedness of other objects by reference to these.[44]

The first option, by contrast, introduces a new inner characteristic with little appreciable gain: namely, one is able to say of a lone hand that it is either right or left. It is generally agreed that such degenerate cases should be decided not by intuitions one might have about the object itself, but on the basis of systematic considerations. Thus, for example, a well known class of curves may be defined as "conic sections". However, some sections of the cone result in straight lines, or a single point. The decision to call these sections (degenerate) circles, parabolas, ellipses, hyperbolas, etc., is not made on the basis of considering the sections themselves, but rather on their relation to the specification of all the other curves. Similarly for the designation of a parabola as an ellipse with one focus infinitely distant from the other. Consider also the decision not to count 1 as a prime number. This is not reached by fixing one's mind on that number, but on the complications which would arise, for instance in the unique factoring theorem, if it were to be called prime.

§.6.

It is one thing to understand the confusion which underlies Kant's thoughts on these matters and another, considerably more challenging, to achieve a plausible account of what he might have been thinking. To this task we now turn.

The existence of incongruent counterparts is nowhere interpreted by Kant as revealing a mathematical error. As strange as it may sound in light of his discov-

[42] Cf. Mühlhölzer, 450.

[43] Cf. H. Weyl, *Symmetry* (Princeton: Princeton University Press, 1952), 21f.

[44] Cf. *Prolegomena*, 4.286, Beck, 34 (emphasis added): "Hence the difference between similar and equal things which are not congruent. . . cannot be made intelligible by any concept, *but only by the relation to the right and left hands. . . .*"

ery, Kant apparently assumed that Leibniz's and Wolff's analysis of the concepts of geometry was entirely correct. In particular, he accepted Wolff's judgment that the only possible external difference between similars was magnitude. This meant that orientation had to be an inner characteristic. However, since Kant also accepted Leibniz's analysis of similarity as definitive, there was no place for this new characteristic in his geometrical system. Having thus precluded a mathematical resolution of his problem, Kant was faced with the problem of explaining the difference some other way. In 1768 he fell back on the only available alternative, Newton's absolute space. This theory *must* explain the incongruence, since Leibniz's theory cannot. But the explanation it provides is somewhat mysterious:

> . . . the complete ground of determination of the shape of a body rests not merely upon the position of the parts relatively to one another, but further on a relation to the universal space which geometers postulate–a relation, however, which is such that it cannot itself be immediately perceived. What we do perceive are those differences between bodies which depend exclusively upon the ground which this relation affords.[45]

By 1770, Kant had another alternative, space as the form of sensibility, which allowed him to reject that "empty creature of reason"[46] which is Newton's absolute space. Again, Leibniz is given full marks for being entirely correct *as far as concepts go*—but the problem is now found to lie in the fact that concepts do not and cannot go far enough: ". . . the difference between similar and equal things which are not congruent [i.e. counterparts]. . . cannot be made intelligible by any concept. . . ."[47] Since concepts alone cannot explain the difference, intuition is pressed into service. As was the case with absolute space, however, the explanation offered by intuition is not very satisfying. It is that there is no explanation–the difference is something that can only be seen:

> . . . this difference admits indeed of being given in intuition, but does not at all admit of being brought to clear concepts and therefore of

[45] 2.381; Handyside, 25-6; cf. 2.377-8, Handyside, 20 (emendations marked with square brackets): "In anything extended the position of parts relatively to one another can be adequately determined from consideration of the thing itself; but the [direction] towards which this ordering [proceeds] involves reference to the space outside the thing; not, indeed, to points in this wider space–for this would be nothing else but the position of the parts of the thing in an outer relation–but to universal space as a unity of which every extension must be regarded as a part."(For a discussion of how the difference between incongruent counterparts can be explained in terms of absolute space, see below, §7).

[46] 2.404.

[47] *Prolegomena*, 4.286; Beck, 33-4. For further consideration of this claim, see below, §8.

> being intelligibly explicated (*dari, non intelligi*)....[48]

The inadequacy of Kant's alternate explanations betrays their origin. The central argument in both cases is a *reductio ad absurdum* of Leibniz' theory of space. Given this, his conclusions are readily understandable once one determines what he thought to be the available alternatives to Leibniz's theory. That is to say, his method was the familiar one of eliminating all possibilities but one, in the hope that that remaining possibility, however improbable, must be the truth. In 1768, Kant was concerned with resolving the Leibniz/Newton dispute. Once Leibniz's theory was ruled out, Newton's was the only remaining possibility, and must, somehow, provide the basis for distinguishing incongruent counterparts. By 1770, the additional alternative allowed the rejection of both Leibniz and Newton.

This procedure cannot inspire much confidence in its results. Apart from the difficulties involved in determining that no further possibilities exist, and the heuristic unpleasantness of apagogical proofs, Kant's method suffers from the problem common to all such arguments, namely, that the number of possible sources of absurdity increases exponentially with the number of premises. This type of argument works best in formal systems of great precision, where all premises and rules of inference but one are reliable. In Kant's argument, as we have already seen, this is far from the case.

§7

What we have so far seen of Kant's reasoning is not the whole of his argument in "Directions." To understand that larger argument we need to address a philological point that has confused nearly all English speaking commentators, and a number of Germans as well.

From the beginning, Kant's argument about counterparts has been vexed by problems of translation. His first English translator, John Richardson, stunningly oblivious to the point at issue, translated "*widersinnig gewundene Schnecken,*" helices of opposite sense, as "snails rolled up contrary to all sense."[49] While this is merely comical, Handyside's mistranslation of "*Gegend* " as "region" in his version of Kant's 1768 essay is serious indeed.[50]

[48] 4.484; Ellington, 23; cf. A.G. Baumgarten, *Metaphysica*, 7th ed. (Halle, 1779; reprint Hildesheim: Olms, 1963), §69.

[49] John Richardson, tr. *Kant's Prolegomena*, (London, 1819). Cited from *Kant's Critical Philosophy,* John P. Mahaffy and John H. Bernard, eds. (London: MacMillan, 1889), II, 39.

[50] Handyside, 1929; the same mistake is made by others: D.E. Walford, in G.B. Kerford and D.E. Walford eds., *Kant: Selected Pre-Critical Writings and Correspondence with Beck* (Manchester: Manchester University Press, 1968), p.36f; also, "Du premier fondement de la différence des régions dans l'espace," trans. S. Zac, in *I. Kant: Quelques Opuscules Précritiques* (Paris: Librarie

We learn from Christian Wolff's *Mathematisches Lexicon* that a *Gegend* (Latin *plaga*) is the intersection of an azimuth with the horizon: "At sea, *Gegenden* are determined with a compass."[51] In "What is Orientation in Thinking," Kant says "We divide the horizon into four [*Weltgegenden*]," that is, main compass points, or quarters.[52] With this emendation, Kant's famous essay becomes "On the First Ground of the Distinction of Directions in Space."[53] Here now is a typical Handysidian sentence: "Even our judgments about the cosmic regions are subordinated to the concept we have of regions in general, in so far as they are determined in relation to the sides of the body."[54] What could this possibly mean? Kant actually said this: "Even our judgments of compass bearings depend upon directions as they are determined by the sides of our body." That is to say, without knowing left from right we could not, given one compass bearing, determine the rest of them: "However well I know the order of the cardinal points, they allow me to determine bearings only if I know towards which hand this order proceeds."[55]

If we look at the 1768 essay with this correction in mind, we note that Kant carries on about directions for two thirds of the essay, and only then comes to the discussion of counterparts. Reconstructions of Kant's argument tend to leave out that first part[56] because, one cannot but conclude, no sense could be made of it. To

---

Philosophique, 1970). S. Körner got it right in *Kant* (Harmondsworth: Penguin, 1955), p. 33. N.B. An anonymous reviewer for *Kant-Studien* has kindly pointed out that this error has been rectified in the recently published translation by R. Meerbote and D. Walford of the 1768 essay in the Cambridge edition of the works of Kant: I. Kant, *Theoretical Philosophy 1755-1770*, D. Walford ed. (Cambridge: Cambridge University Press, 1992).

[51]Christian Wolff, *Mathematisches Lexicon*, (Leipzig 1716; reprinted Hildesheim: Georg Olms, 1965), p. 659. The same in Johann Christian Adelung, *Wörterbuch der Hochdeutschen Mundart*, (Leipzig 1796; reprinted Hildesheim: Georg Olms, 1979), Pt. 2, 481, and Grimm, *Deutsches Wörterbuch*, Vol. 4, pt. 1, 2230. Kant himself frequently uses "*Gegend*" in the sense of "direction". See especially the appendix "On Navigation" to the *Physical Geography* (9.307). Cf. also the essay on Negative Quantities (2.171), where it is used synonymously with "Richtung".

[52]8.134.

[53]This error has led to deep, if misguided, speculations about Kant's concept of a region: A.T. Winterbourne, *The Ideal and the Real*, (Dordrecht: Kluwer, 1988), 70-72, Peter Alexander, "Incongruent Counterparts and Absolute Space," *Proceedings of the Aristotelian Society* 85, 1985, 1-22; William Harper, in Van Cleve and Frederick (1991) 292 ff., Jill Vance Buroker, *Space and Incongruence,* (Dordrecht: Reidel, 1981), 57. The mistake was not, however, just a problem of translation. "*Gegend*" now usually does mean "region", leading at least the following to fall into this trap: H. Vaihinger, *Commentar zu Kants Kritik der reinen Vernunft*, II, 522,526, where *Gegend* is evidently interpreted as *subspace*. Werner Gent, *Die Philosophie des Raumes und der Zeit* (Hildesheim: Olms, 1962), I, 271. Gent's use of scare quotes indicates his discomfort with this reading. Most recently, P. Janich, *Euclid's Heritage: Is Space Three Dimensional?* (Dordrecht: Kluwer, 1992), 29f.

[54]Handyside, 29; 2.379.

[55]*Ibid.*

[56]E.g. in Van Cleve and Frederick: Nerlich, 153; Van Cleve, 204; Earman, 235 f.; Harper, 263 ff.; Buroker, 323 f.; Van Cleve, 344 f.

be sure, there are interesting problems in the latter part, but the full force of Kant's snappy argument is lost, namely, were Leibniz's theory of space true, we would not only be unable to distinguish our left and right hands, but also to navigate a ship, or orient a book so that it might be read.

Kant's remarks suggest that if space has absolute directions, then hands are left and right because of their relations to these directions. For example, if the index finger points (absolute) up, the middle finger (absolute) north, and the thumb, at right angles to both of them, (absolute) west, then the hand is a right hand. Rightness thus "depends upon" absolute direction. Kant thus sides with those followers of Pythagoras who, as Aristotle informs us, held that there is a left and a right in the heaven.[57] The reader should note that if there are absolute directions, then this is a perfectly reasonable way of specifying orientations, even if it is open to Leibnizian objections. Kant's mistake is the usual one: he thinks that a sufficient condition he has found is the only one that can be found, and is thus necessary. This is sometimes called an inference to the best explanation and, indeed, an absolute space with absolute directions was, by Kant's lights of 1768, the best explanation available.

Ironically, in the earlier part of his essay Kant had, in effect, taken the steam out of this line of argument. In "Directions" and again in "Orientation", Kant considers the human ability to determine bearings, and asks his familiar question: How is this possible? The answer is relatively straightforward. Space is three dimensional, and thus it is possible to imagine three mutually perpendicular planes in it. Taking advantage of obvious asymmetries, one of these planes may be conceived so that it divides the body into two parts, above and below; a second divides it into front and back.[58] The third plane divides the body into two almost symmetrical halves, left and right.

Although very nearly symmetrical, the left and right halves of the body are nevertheless readily distinguished. For, Kant notes, the right hand is (generally) stronger and more dexterous, the left hand more sensitive. Other asymmetries are noted; for instance, the presence or absence of a heartbeat is sufficient to distinguish the two sides. Also, the viscera are arranged asymmetrically, hops turn from left to right around the pole, beans from right to left; the hair on the crown of the head swirls from left to right, and so forth. The asymmetries of the body are

[57] *De Caelo*, II,2.

[58] Although the distinctions front/back and above/below are noted by Kant, he sometimes forgets them. Thus, for instance, in the *Metaphysical First Principles of Science,* he writes that motion in a circle may be in one of two directions (4.484). This is false, for the motions are congruent. Kant probably assumes that, in addition to the plane of the circle, a distinction of above and below (i.e. on lines normal to the plane) is given. Cf. his remarks on the orientation of a written page (2.379). For a similar oversight, cf. Aristotle, *De Caelo*, II,5 (287^b22f.).

accompanied by a "feeling" of left and right:

> Since the difference in feeling of the left and right side is of such great necessity for the judgment of directions, nature has connected it with the mechanical arrangement of the human body.[59]

Shortly afterwards he remarks that

> . . . the two sides of the human body, in spite of their great outer similarity, are sufficiently distinguished by a well-marked feeling, even if we leave out of account the differing positions of the inner parts and the noticeable beat of the heart. . . .[60]

Given that we are able to distinguish left from right (also above from below, front from back), the business of orientation is straightforward. Suppose "north" to be determined; if one faces it (with one's body perpendicular to the surface of the earth and one's feet on the ground), then "east" is the next cardinal point to the right.[61]

How does this connect with incongruent counterparts? Evidently, Kant was a firm disbeliever in parity. The violation of parity in particle physics, much talked about lately and sometimes advanced as providing a solution to a Kantian problem (but which?), would not have surprised him in the least. We have given part of his long list of asymmetrical objects, which "nature has arranged." These asymmetries, whether external or in our body, teach us to tell left from right. And since the difference between counterparts is the "direction towards which their parts are ordered,"[62] our ability to make this distinction allows us to *apprehend* the incongruence of certain similar and equal bodies.

Note, however, that this does not give us a grasp of absolute "rightness" or "leftness" but only of "having the same orientation as the strongest hand of most people"–i.e. a property rooted in a comparison. That is, the human ability to distinguish left from right does nothing to establish that handedness is an inner characteristic. Hence this part of the essay suggests a solution of the right-left problem independent of absolute directions. We do not claim, of course, that absolute directions are inconsistent with violations of parity, only that if the latter are as far-reaching and palpable as Kant suggests, then the argument for absolute directions loses its force: it was to be an inference to the best explanation, but violations of parity are, surely, just as serviceable.

[59] 2.380, Handyside, 24.
[60] 2.381; Handyside, 25.
[61] Cf. 2.379.
[62] 2. 377.

§8

We now return to Kant's statements, in the Dissertation and later, that concepts alone are unable to describe the difference between incongruent counterparts. We shall not consider this claim, which is notoriously absent from the First Critique, in its role as a support of transcendental idealism, but merely its plausibility.

Kant's claim concerning the inadequacy of concepts is clearly intended to be an absolute one–i.e. he does not want to make the point that, relative to some set of concepts (e.g. {"flat", "curved", "made of cheese"}), incongruent counterparts are indistinguishable. But if intended in an absolute sense, it would seem to require proof. None is offered, and there is not the slightest indication that Kant saw the necessity for one. The reason for that has already been mentioned: to say that incongruence is explainable on the basis of concepts means, for Kant, that we already have these concepts, and that their *analysis* provides the desired explanation. In other words, Kant thought he (and we all) had, roughly as provided by Euclid, all basic geometrical concepts to hand. Otherwise, we must assume Kant to be making a prophetic and wildly speculative declaration about the future development of mathematical concepts, something which does not square with his matter of fact tone. Kant's views on the completed state of Logic are well known,[63] and it seems reasonable to suggest that he held similar views with regard to basic geometrical concepts.

In particular, since "congruence" is revealed by analysis to mean "similar and equal", the difference between equal but incongruent figures should be explainable by concepts falling under Kant's concept of similarity. That this is not possible is a point which must be, and has been, granted. It is quite another thing, however, to claim that *no* concept can express the difference. Consider for a moment the strangeness of Kant's assertion that "...the difference between similar and equal things which are not congruent...cannot be made intelligible by any concept." This statement is one which refutes itself as soon as it is uttered, as Bolzano once remarked concerning a similar deliverance of the critical philosophy.[64] For, clearly, with the concepts "similar", "equal", "congruent", "and", "not" (as he understood them), Kant has precisely described the difference.[65] This point, which seems trivial, evidently did not appear so to Kant. Why was this so?

[63] E.g. *Critique of Pure Reason*, B viii.

[64] B. Bolzano, *Wissenschaftslehre* (Sulzbach, 1837), §79, note γ.

[65] That is, given any non-symmetric figure in Euclidean 3-space, there exists a unique figure which differs from it in the way Kant describes (i.e. by being similar and equal but incongruent to it). By "*unique* figure", we mean that any two point sets which satisfy the stated condition are congruent to each other, i.e. "figure" denotes an equivalence class of loci under the relation "congruence".

One prominent possibility centres on Kant's understanding of congruence. The basic meaning of this notion of course stays the same: two figures are said to be congruent if capable of being moved to coincide. Kant's assumes considerably more than this, however, for congruence is also employed as the geometrical version of identity.[66] And, given this assumption, just as it is insufficient to say that two things could differ *only* in being non-identical (for there must always be some specifiable difference), so too it might seem illicit to stipulate that two geometrical figures could differ *only* in being incongruent.

Now if one assumes, as Kant apparently did, that all possible differences between geometrical figures can be accounted for by the predicates of Leibnizian similarity and equality of magnitude, then the only difference between counterparts *is* incongruence. Kant's description of the difference between counterparts therefore reduces to the statement that such pairs of objects differ only in being non-identical. And this is clearly unacceptable if offered as a conceptual account of the difference.

Although one can follow this line of thought, one has great difficulty approving of it. If, as Kant suggests, congruence is to be interpreted as identity, then counterparts point to a defect in the list of inner and outer characteristics—i.e. there are differences which cannot be accounted for by means of these. Kant gave a name (*Gegend* or *Seite*) to the missing characteristic but, strangely enough, persistently denied that it was a fully fledged concept. It is, he says, "a concept that can be constructed, but in no way can be made clear as a concept in the discursive mode of cognition"(4.484). What Kant claims to be constructing here is not "left" or "right", but "having the same orientation as (a paradigm)", e.g. "with the sun" or "against the sun" (*ibid.*). It is this sameness of orientation that cannot be expressed in concepts, he claims. But, as noted, he only scanned his existing repertoire, in terms of which orientation is indeed undefinable.

Even supposing we follow Kant in his conceptual conservatism, however, and do not introduce "orientation" as a new primitive, we can, as just mentioned, precisely describe the difference between counterparts, provided we can use "congruence" as a concept (in the expectation that its difference from simple identity will be sorted out later). Apart from the assumption that the list of mathematical primitives is already complete, there is no reason, even in Kant's understanding of mathematics, to deny congruence this status; it is, after all, a concept that can be constructed, and in this sense fits in with the rest. In other words, one need not subscribe, for example to Weyl's combinatorial account of the matter, but can have congruence as a concept without breaking faith with Kantianism.

[66]Cf. Dissertation, 2.403, where incongruence is equated with non-identity (diversity). Also, 4.285.

To sum up: through his acquaintance with counterparts, Kant had stumbled upon a problem which to his knowledge had not been adequately dealt with by previous geometers, and had a variety of means at his disposal to solve the problem purely mathematically. That he did not do so must be attributed to his belief in the completeness of existing mathematics. His recognition of incongruent counterparts, on this view, would not be understood by him as the discovery of the need for conceptual changes in the geometrical systems he knew, but rather of something beyond the reach of all concepts. Thus, instead of pointing out an error in Euclid and Leibniz, and looking to mathematics for a solution, Kant decided that the difficulty was rooted in a philosophical assumption. Put another way, Kant had too much respect for the mathematical acumen of others, and not enough for his own.

§**9**

Kant was not the first, and he has certainly not been the last, to run into difficulties in discussing the problems of left and right. Aristotle, for one, following Pythagorean precedent, attempted to induce distinctions of up/down, front/back and left/right on the celestial spheres and on most of animate nature as well.[67] The results of this endeavor are not an unqualified success–the leaves of plants, for instance, must be said to be below the roots.[68] We have seen that Euclid, and after him Leibniz, failed to take account of incongruent counterparts in their definitions of similarity, and thus fell into error. Kant has the merit of having noticed their oversight, but his reactions to this discovery seem to us somewhat eccentric.

His discussion of the problems of orientation and incongruent counterparts rests upon a small mountain of terminological and scientific confusion. Once this has been sorted out, his problem disappears. We cannot therefore agree with Mühlhölzer, when he writes that Kant cannot be faulted for failing to have solved the paradox of incongruent counterparts since the problem required advances in logic and mathematics not available until the present century:

> Of course, no criticism of Kant arises from [our] increased knowledge. Neither the deductive nor the conceptual resources of modern logic and mathematics were available to him. He saw clearly that his logic could not do justice to mathematical (and especially geometrical) knowledge: neither to the conclusions drawn in Euclidean geometry,

[67] *De Caelo*, II,2 (284^b6f.); cf. *De Incessu Animalium*, 4 (705^a26f.).

[68] Heath, *Aristarchus of Samos* (Oxford: Clarendon Press, 1913), 231, who gives a detailed exposition of Aristotle's attempts, accurately describes them as "not important, but. . . not unamusing."

> nor to the concepts required for a complete grasp of the facts [*Sachverhalte*] of geometry. Kant used intuition to fill these gaps, a move which was entirely justified at that time.[69]

This assessment does not sit well with the fact that Kant succeeded, albeit unwittingly, in precisely stating the difference between incongruent counterparts purely by means of concepts available to him. Moreover, shortly after Kant had stated his problem, two mathematicians gave the outlines of straightforward solutions, Bolzano by pointing out that orientation (*Gegend*) is an external characteristic like magnitude, Gauss by noting that, while Euclidean 3-space is orientable, the assignment of names to orientations (e.g. "right handed coordinate system", etc.) is conventional and must be communicated by ostention.[70]

One might think, and it has been suggested,[71] that communication "by ostention" requires intuition after all, and that this somehow vindicates Kant. But what is meant here? We should note, first of all, that it is not suggested, and even Kant never believed, that we convey by ostention, or can have an intuition of, "left" or "right" as such: "The two sides externally in intuition show no difference that we can notice" (8.135). We are, however, "enabled by nature and skilled by exercise" (*ibid.*) to pick out left and right through a feeling which may have its origin in nature's relentless violations of parity.

Rather, the point of ostention is not so much a recourse to intuition, but to comparison. This can be made clear through another external characteristic, size. Suppose that on some planet there is a culture much like ours, except that they measure length with a unit which they call an "inch". No discursive information relayed via, say, a radio signal can convey to us how many inches there are in a centimetre. We must be pointed to a paradigm, as which the radio signal itself may (with some further assumptions) be made to serve. All other lengths can then be determined by comparison with this. This is not in the least surprising, if indeed size is an external characteristic.

For reasons not wholly obvious this problem of communication has attracted much less attention than the analogue that arises in connection with the external characteristic orientation.[72] Leftness designates one of two classes of incongruent

[69]Mühlhölzer, 452.

[70]B. Bolzano, *Betrachtungen über einige Gegenstände der Elementargeometrie*, §24; C.F. Gauss, "Theoria residuorum biquadraticorum, comm. secundo," *Werke* (Göttingen, 1863; reprint Hildesheim: Olms, 1975) II, 177. Also, H. Weyl, *Philosophy of Mathematics and Natural Science* (New York: Atheneum, 1963), §14; F. Klein, *Elementary Mathematics from an Advanced Standpoint* (New York: Dover, 1948), p.39-42; and A. Grünbaum, *Philosophical Problems of Space and Time,* 2nd ed. (Dordrecht: D. Reidel, 1973), p. 331-2, give brief solutions to Kant's problem.

[71]By an anonymous reviewer for *Kantstudien*.

[72]The latter has been called the "Ozma" problem by Gardner.

objects. Which one it is must be conveyed again by an archetype or paradigm to which we are pointed, and in comparison with which other objects are seen to be either of the same, or of the opposite orientation. All that intuition does is a matching up of objects. This is not a specifically Kantian point, and it is not in the least surprising: if someone wants me to sort a pile of gloves into pairs, I must of course look at each of them to carry out the matching.

Now the transfer of the paradigm is no more of a mathematical act than the depositing of the standard metre in Paris. When Gauss lets ostention serve as the vehicle for conveying the choice of left and right, he does not, after all, admit intuition into mathematics by a back door. It does not belong there, and none of Kant's arguments show that it does.

§**10**

It is difficult to understand, at first glance, why Kant did not expect to find a mathematical resolution of his problem. Instead of seeking the source of his "paradox", he willingly accepts it. Hermann Weyl had no patience for this procedure:

> Kant finds the clue to the riddle of left and right in transcendental idealism. The mathematician sees behind it the combinatorial fact of the distinction between even and odd permutations. The clash between the philosopher's and the mathematician's quest for the roots of the phenomena which the world presents to us can hardly be illustrated more strikingly.[73]

In a later work, Weyl went still further, and suggests an antiscientific inclination as the cause:

> Scientific thinking sides with Leibniz. Mythical thinking has always taken the contrary view as is evinced by its usage of right and left as symbols for such polar opposites as good and evil. You need only think of the double meaning of the word right itself.[74]

These attacks are excessively unkind: Kant's credentials as a scientist and debunker of mysticism are not inconsiderable. Kant's expectations were shaped by what he knew and understood of the mathematics of his day. Euclid's geometry, with its appeals to intuition and construction, was still widely considered a preeminent model of rigor. Leibniz's project to conceptualize geometry, as far as Kant knew, "never existed save in intention"[75] and, from his remarks on Wolff's definition of similarity [76]one may surmise that he thought this project to have been

[73]*Philosophy of Mathematics and Natural Science,* 84.

[74]*Symmetry*, 21-22.

[75]2.377.

[76]2.277, quoted above, p. [6]; cf. *Critique of Pure Reason*, A727/B755: "... in mathematics the employment of a philosophical method results only in mere talk."

profoundly misguided. Decisive results of conceptualization in the infinitesimal calculus would not be achieved for several decades, with geometry lagging somewhat behind. Guided by intuition, mathematics seemed much more secure to Kant than philosophy with its troublesome concepts.[77] That conceptualizations of geometry could fall short, therefore, might not have seemed at all unusual to him. This would not be of overriding concern to him in any case, since he thought that geometry had a perfectly solid non-conceptual foundation.

[77]See especially, the "Inquiry on the clarity of the principles of natural theology and morals," 2.282f. One of the subheadings of this work is. "Das Object der Mathematik ist leicht und einfältig, der Philosophie aber schwer und verwickelt."(2.282)