The *Beyträge* at 200: Bolzano's quiet revolution in the philosophy of mathematics

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Abstract: This paper surveys Bolzano's *Beyträge zu einer begründeteren Darstellung der Mathematik* (*Contributions to a better-grounded presentation of mathematics*) on the 200th anniversary of its publication. The first and only published issue presents a definition of mathematics, a classification of its subdisciplines, and an essay on mathematical method, or logic. Though underdeveloped in some areas (including, somewhat surprisingly, in logic), it is nonetheless a radically innovative work, where Bolzano presents a remarkably modern account of axiomatics and the epistemology of the formal sciences. We also discuss the second, unfinished and unpublished issue, where Bolzano develops his views on universal mathematics. Here we find the beginnings of his theory of collections, for him the most fundamental of the mathematical disciplines. Though not exactly the same as the later Cantorian set theory, Bolzano's theory of collections was used in very similar ways in mathematics, notably in analysis. In retrospect, Bolzano's debut in philosophy was a remarkably successful one, though its fruits would only become generally known much later.

Introduction

Two hundred years ago in Prague, a little-known professor of religion at the Charles University published a manifesto calling for a revolution in mathematics and logic, promising himself to undertake a complete reconstruction of mathematics from the ground up, in accordance with the principles of his new logic, itself a work in progress. Bernard Bolzano (for this was the professor's name) had no gift for catchy titles. He called his book *Contributions to a Better-founded Presentation of Mathematics* (Bolzano 1810). The *Contributions* were intended to be but the first installment of a series of publications setting out Bolzano's views on the foundations of the various branches of mathematics, both pure and applied. Lack of interest in the first issue, however, led him to postpone this project, and instead to first publish a series of papers he thought more likely to catch the public's attention. These papers are to the first installment what Descartes' *Geometry, Optics,* and *Meteorology* are to the *Discourse*, namely, samples of the promised fruits of the method. Among them are a pair of papers on the foundations of real analysis which achieved decisive results, as well as a more speculative paper which, among other things, contains suggestive fragments of point-set topology (Bolzano 1816, 1817a, 1817b). Though these papers did not become as famous as Descartes's *Geometry*, the *Purely analytic proof* was perhaps just as influential, finding avid readers, notably in the circle of mathematicians around Weierstrass, and, as has been argued elsewhere, indirectly influencing through them the development of analytic philosophy (Rusnock 1997a).

Though no spring chicken, Bolzano in 1810 was still a novice in philosophy, particularly in logic. And it shows—throughout the *Contributions*, we find traces of half-digested theses taken from the logical literature of the time. Like many of his contemporaries, Bolzano uses the term 'judgment' to refer indifferently to the individual, subjective *acts* of judgment or to the objective *content* of these acts.[1] He adopts Kant's definition of analytic judgment, while at the same time agreeing with Locke that such judgments are 'trifling' (Bolzano 1810, II, §§17–18) and even, in what he would surely later recognize as a moment of confusion, suggests that the most general concept is that of an *idea* (1810, II, §5). Though there is some tinkering around the edges, Bolzano's treatment of formal logic is quite traditional. Judgments are held to be all of the subject-predicate form (though he proposes that several different copulae should be recognised) (1810, I, §15). More limiting was his view on the forms of subject- and predicate-terms, which, in line with tradition, he seems to regard as simple sums of (positive or negative) characteristics.[2] Although he takes a timid step beyond the confines of traditional syllogistic, the forms of inference he enumerates are limited in the extreme, especially in comparison to the infinite variety he would later recognise (Bolzano 1810, II, §12; 1837, §155).

In light of Bolzano's later logical discoveries, it is somewhat surprising to find that in 1810 his formal logic reflects, by and large, the conventional wisdom of the day. But there was a good reason for this, namely, that he had put most of his energy at first into the study of mathematics. In the decade following the publication of the *Contributions*, he made a series of mathematical breakthroughs, discovering how epsilontics could be used to provide a foundation for the infinitesimal calculus and the theory of power series, developing elements of point-set topology for use in geometry, and investigating the theory of collections as the most fundamental mathematical discipline. These mathematical discoveries must have made plain the inadequacy of the logic he had merely sketched in the *Contributions*. In any case, during the decade of the 1820s, Bolzano worked steadily at logic, completely revising his earlier views.

[1] He would later regard his earlier usage as ambiguous, coining the term 'proposition in itself' to refer to the contents of (possible) judgments. See Bolzano 1837, §§19 ff.

[2] This seems to us to follow from Bolzano's claim that the general forms of all definitions are '$(a\ cum\ \alpha) = A$' and '$(a\ cum\ non\ \alpha) = A$'; see Bolzano 1810, II, §3.

For all its flaws, the *Contributions* nonetheless contains the seeds, and many of the motivating ideas of these later developments. What it lacks in technical sophistication it more than makes up for in spirit and vision. It brings the following innovations :

- a new definition of mathematics and a new organisation of its subdisciplines,
- the statement of the objectivity of mathematics,
- one of the first, if not the first, modern studies of axiomatics,
- a presentation of logic as an integral part of axiomatics,
- in an appendix, a criticism of Kant's philosophy of mathematics, written by a philosopher who is at the same time a working mathematician,
- and in the second issue, unfinished and published only in 1977, a first draft of a theory of collections under the title *Mathesis universalis* (*allgemeine Mathesis*).

The first issue consists of two parts: I) on the concept and division of mathematics, II) on the mathematical method. For Bolzano, a presentation of the mathematical method is essentially nothing other than logic and not itself part of mathematics; in fact, it precedes mathematics. The title of the *Contributions* reveals the importance that Bolzano attaches to the rules of exposition of sciences. All his works on logic exhibit the same structure and the same headings: the last paragraphs of *On logic* (Bolzano 1977b) as well as *On the Mathematical Method* (Bolzano 1975a) recall these rules, by means of which Bolzano will define the ultimate goal of the theory of science: to present the rules "we must follow in dividing the total domain of truth into individual sciences and which must govern the writing of their respective treatises"—in brief, "it is the science which instructs us in the presentation of sciences in well-constituted treatises" (Bolzano 1837, §1) The reader recalls Pascal's art of persuasion, which consists in "the conduct of perfect methodical proofs" (1963, 356).

Following Kant, in 1810 Bolzano rejects the traditional definition of mathematics as science of quantity, because many mathematical propositions, e.g., in combinatorics or geometry, do not concern quantities (1810, I, §3). But he strongly disagrees with Kant, who based mathematics on constructions of concepts in pure intuition (= in the forms of intuition that are time and space); according to the latter, such constructions are the warrant for the existence of mathematical objects. For Bolzano, Kant's notion of a pure intuition, which is supposed to be at the same time singular and universal, is simply contradictory, and neither universal mathematics nor even geometry requires intuition (Bolzano 1810, I, §6; appendix, §§8-9). Constructions are just illustrations of theorems and proofs, important for pedagogical reasons and for facilitating better insight into the details of arguments, but rigorously scientific proofs must be conducted by purely conceptual means. Contrary to Kant, for whom mathematical objects exist only when they are constructed, for Bolzano, the only thing required for the existence of mathematical objects is the compatibility of the concepts that define them (Bolzano 1977a I, §25; cf. Bolzano 1837, §352).

What, then, is mathematics? It is a "science that treats general laws (forms) to which things must conform in their existence" (Bolzano 1810, I, §8). Bolzano adds that his definition applies to all things whatsoever, both physical and purely mental, such as intuitions and ideas (representations). An example of such a form is countability, according to which it is possible to compose or unite equal parts of a thing. Later formulations indicate that mathematics studies the conditions of possibility of things, e.g., all possible configurations of a collection of objects, while metaphysics has "the single concern of proving, from *a priori* concepts, the actual existence of certain objects ... such as freedom, God, and the immortality of the soul" (1810, I, §9). Mathematics and metaphysics are thus two branches of *a priori* knowledge: mathematics also considers actually existing things, but deals only with hypothetical necessity, while the task of metaphysics consists in proving the necessary existence of some of these things.

This new definition of mathematics did not escape the attention of Husserl.[3] Already in the *Logical Investigations*, he had praised Bolzano as "one of the greatest logicians of all times", whose *Theory of Science* "far surpasses everything that world literature has to offer in the way of a systematic sketch of logic" (Husserl 1970, Vol. 1, 222.) Having read the second edition of the *Contributions* in 1926, Husserl noted the kinship of his own definition of formal ontology with Bolzano's definition of mathematics:

> Precise inspection shows that here Bolzano gives a definition (which needs improvement, to be sure) of a universal *a priori* ontology that comprises both a material and an empty-formal ontology, without drawing a distinction between them. He then attempts, it is true, the isolation of a "universal mathematics", in which "the theory of numbers, the theory of combinations, etc." are to be included. He emphasizes that disciplines such as geometry and chronometry must be considered, not as coordinated with those, but as subordinate to them; and he finds the distinguishing characteristic of the former disciplines in

[3]For discussion of Bolzano's influence on Husserl, see Centrone, 2010.

> the circumstance that their laws “are applicable to all things without exception,” while the other disciplines are not (Husserl 1974, §26 d).

According to Husserl, Bolzano conflates two concepts of ‘thing in general’: on one hand, it is simply the concept of the ‘empty formal form of something in general’, and on the other the universal concept of reality which differentiates into particular regions with their particular ontologies. But his main objection to Bolzano is the absence of formalization, notably the lack of an adequate treatment of formal logic by means of a symbolic language. In the end, he judges, “Bolzano did not attain the proper concept of the formal, the concept that defines formal ontology, although in a certain manner he touched upon it” (Husserl 1974, §26 d).

Bolzano’s definition of mathematics yields a new way of organising its disciplines. According to the Euclidean scheme, there are two branches of mathematics: arithmetic and geometry. For Bolzano, these are not on the same level, because their objects do not have the same generality. The object of pure or universal mathematics, *mathesis universalis*, is the thing in general [*Ding überhaupt*]; its laws apply to all things without exception. Geometry, by contrast, becomes part of applied mathematics; it deals with the smaller domain of objects that are in space. Disciplines which only acquired a well-defined status in the seventeenth and eighteenth centuries, such as algebra, combinatorics and the infinitesimal calculus, along with arithmetic, are now recognized as basic disciplines yielding concepts applicable throughout mathematics. This reversal of the traditional order and the reclassification of geometry creates the need to devise new proofs for old theorems: everywhere that geometrical proofs of theorems of analysis were considered satisfactory, new, purely analytical proofs and new concepts belonging to pure mathematics will be necessary. Gauss and Bolzano would be the leading figures in this movement to arithmetize analysis, followed by Cauchy, Abel and others.

The objective order

The prevailing views on mathematics among philosophers of Bolzano’s time, notably Kant and his followers, differed slightly if at all from those put forward a century and a half earlier by Descartes and Pascal (Descartes 1985a, 1985b, part 2; 1985c, I, §13; Pascal, 1963; Kant, 1992, part 1; Kant 1996, A712/B741 ff.). If anything, some of the finesse of these authors may have been lost through transmission. Proof was viewed primarily in subjective terms: a proof is a series of considerations through which the truth of a judgment becomes evident. Obviously, it seemed, if a proof is to succeed, its premises must themselves be evident. This may occur because these premises have themselves been proved, but clearly this cannot proceed *ad infinitum*. There must thus be some judgments which are evident in and of themselves to serve as the starting point of proofs, and these are the principles, axioms, postulates, or common notions. Definition, too, was conceived in subjective terms, its purpose being to bring us to a clear understanding of a concept or of the meaning of a term. Obviously, again, this will only work if the terms used in the definition are themselves clearly understood. Now this may occur because those terms have themselves been defined, but if we are to avoid an infinite regress or a circle, there must be some terms which are clearly understood in and of themselves, and these are the primitive or indefinable terms.

Mathematics, at least in its ideal form, would be structured as follows. We would begin with a set of terms clearly understood in and of themselves, defining all others in terms of these. We would then set out a number of truths which are evident in and of themselves, and prove all others from this initial stock of premises.

Near the beginning of the second part of the *Contributions*, entitled "On the mathematical method", Bolzano makes it clear that he rejects this conception:

> This much . . . seems to me to be certain: in the realm of truths, i.e., in the collection of all true judgments, there is an objective connection, independent of our subjective recognition of it; and that, as a consequence, some of these judgments are the grounds for others, and the latter the consequences of the former (1810, II, §2).

If we follow Bolzano here, we will have to recognise that, independently of human minds and their capacities, activities etc., some truths, called *consequences*, are dependent upon others, their *grounds*. If, in addition, we suppose with Bolzano that some truths are basic, having consequences but no grounds, there will also be objective notions of *axioms* or *principles* [*Grundsätze*] on the one hand and *theorems* on the other. Truths will have this status in and of themselves, and not relative to how anyone came to know them or may have presented them in some treatise or other. Similar things may be said about the *definitional* order of mathematics: objectively, regardless of whether anyone is aware of it, some concepts are composed of others, and some concepts are indefinable, not from the subjective, human point of view but in and of themselves.

This is a declaration of war on Kant and all subjectivist philosophers, for whom truths and especially scientific and mathematical truths are the work of man, having their origin in the particular capacity of the human mind that organises our perception and knowledge. Surprisingly, though, Bolzano does not oppose the subjective order (*ordo cognoscendi*) to the order of being (*ordo essendi*) as the Scholastics, following Aristotle, had. Rather, Bolzano's is a third way: the objective order he has in mind is one that is intrinsically logical, not dependent on epistemology, metaphysics, or anything external to logic. Moreover, it is fixed neither by God's will (as Descartes had maintained) nor even by God's understanding (as Leibniz had claimed); rather, as he would later write, God can only recognise and accept it:

> I agree completely with Leibniz when he claims that the truth of laws and Ideas does not depend on God's will ...; but when he adds that the necessary truths depend only on God's understanding, I contest this as well, and indeed affirm the exact opposite. It is not the case that $2 \times 2 = 4$ because God thinks so; rather, because $2 \times 2 = 4$, God thinks so (Bolzano 1979, 44).

The doctrine of the objective connection of truths is the seed from which all of Bolzano's logical theories will grow. It means that the real purpose of a scientific exposition is the organisation of truths according to their relations of objective dependence, and not simply certainty and strength of conviction.

Axiomatics and scientific presentation

The *Contributions* contain one of the first modern studies of axiomatics, including accounts of definition, principles (axioms), inferences, and proof. Once again, Bolzano finds himself opposed to Kant in maintaining that construction and intuition play no role in axiomatic mathematics. Rather than having its own method, mathematics employs the same method as philosophy, which is simply logic (1810, II, §1). Though Bolzano was not yet aware of just how inadequate contemporary formal logic (including his own) was for this purpose, he could already see enough to reject Kant's claim that logic had been a complete and perfect science since the time of Aristotle (Kant 1996, A xiv, B viii). In the *Contributions*, there is already a trickle of innovations; the *Theory of Science*, written in the 1820s, would bring a flood.

On the methodological side, Bolzano has already made crucial distinctions which would guide his mathematical research for the rest of his life. In the case of definitions (1810, II, §§3–8) to begin with, he thinks we must carefully distinguish:

- bringing someone to a clear understanding of a term/concept,
- conveying the meanings of signs, and
- propositions stating the constituents of a complex concept and their manner of combination (this is what Bolzano calls a definition [*Erklärung*] in the strict, objective sense).

These are clearly very different things, and a concept that requires definition in the third sense (i.e., a complex concept) may not require it in the first sense (i.e., it might already be clearly understood in and of itself). Bolzano's favorite examples are geometrical concepts such as line, surface, and solid. Another good example is the concept of continuity in analysis, the sort of concept which, as Bolzano would put it "everyone knows and doesn't know", and which he defined in the 1817 paper *Purely Analytic Proof* (Bolzano 1817a, preface, II, a). Even if these concepts are thoroughly familiar, they may still be complex, and require definition in the third sense. Moreover, the objective definition may not be immediately helpful in bringing others to a clear understanding of a concept, as anyone who has taught calculus can attest is the case with the concept of continuity and many other, similar ones.

At the same time, there is no guarantee that a concept that neither requires nor admits of definition in the third sense (i.e., a simple concept, one with no parts) will be clearly understood in and of itself. Indeed, Bolzano thinks that it can be quite difficult to grasp simple concepts in isolation, and to become clearly aware of them. They may be better known in the sense that one must think them in order to think the complexes of which they are parts, but they need not be better known in the sense of being more familiar, or even being designated by their own words in ordinary language.

The second kind of definition, those that convey the meanings of signs or symbols, includes what we now call explicit definitions. But this means is not always available, especially when we attempt to give definitions that respect the objective order. What, in particular, can be done to convey the meanings of signs that designate simple, primitive concepts? Bolzano discusses this problem in II, §8 of the *Contributions*. One of the most important methods, which he calls *circumscription* [*Umschreibung*], is described as follows:

> . . . he assists them by stating several propositions in which the concept to be introduced occurs *in different combinations* and is designated by its own word. From the comparison of these propositions, the reader himself then abstracts which particular concept the unknown word designates. So, for example, from the propositions: *The point* is the *simple* in space, it is the *boundary* of a line and itself no *part* of a line, it has neither extension in length, breadth, nor depth, etc., anyone can gather which concept is designated by the word 'point'. This is well known as the means by which we each came to know the first meanings of words in our mother tongue (1810, II, §8; cf. the later formulations in 1975a, §9, no. 1 [p. 67]; 2004b, 57-58; cf. 1837, §668, no. 9).

Several years after the *Contributions*, others—J. D. Gergonne (implicit definitions), Jeremy Bentham (paraphrases), Poincaré (definitions in disguise) and Hilbert (axioms define the primitive terms), to name just the most important—would present similar theories which, despite Frege's and Russell's objections, would become standard axiomatic procedure. This is not to say that Bolzano's view agrees with that of, e.g., Hilbert in all respects. For him, underlying any genuine system of signs is a collection of objective concepts. Circumscriptions do not create meanings *ex nihilo*—rather, they permit readers to latch onto meanings that are already there.

Similar distinctions apply in the case of proof. Here, Bolzano distinguishes proofs whose aim is to convince us that a proposition is true (which he calls *certifications* [*Gewissmachungen*]) and those which indicate the objective grounds of a given truth (proofs in the strict, objective sense, which Bolzano calls *objective groundings*). While it is possible for a single proof to fulfill both of these functions, Bolzano thinks it obvious that in many cases certifications are anything but objective groundings. Think, for example, of the long calculations Newton performed to verify the binomial theorem. Though these did much to strengthen his conviction in the correctness of the formula he had discovered, they can in no sense be looked upon as proving it objectively.

Mathematical proof in the second, objective sense ultimately begins with basic propositions or axioms [*Grundsätze*] in Bolzano's view, but what counts as an axiom cannot be decided on the basis of evidence or certainty. For evidence is subjective, admits degrees, depends on circumstances, and varies from one person to the next, while the axioms Bolzano is interested in have this status objectively. On this point, he invokes the authority of Euclid, who undertook to prove even the most evident theorems when he was able to. He conjectures that Euclid stated his parallel postulate as a postulate only because he did not know how to prove it. An axiom in the objective sense is "a truth which not only we do not know to prove, but which is in itself unprovable" (1810, II, §11).

Clearly, there is no need to prove something in the first sense to someone who is already certain of its truth. Yet a perfectly obvious proposition may nonetheless require proof in the objective sense, as Bolzano claimed in his first publication:

> . . . the obviousness of a proposition does not free me from the obligation to continue to search for a proof of it, at least until I clearly realize that absolutely no proof could ever be required, and why (Bolzano 1804, preface).

On the other hand, there is no reason to suppose that a proposition which is unprovable in the objective sense will be evident. Axioms [*Grundsätze*] may thus require proof in the subjective sense, and these proofs will perforce make use of truths that are, objectively speaking, their consequences. Thus it seems that Bolzano would have no objection to saying that in such cases the consequences are used to prove their grounds (subjectively), and the grounds in turn used to prove the consequences (objectively). Once we have distinguished two different senses of proof, such a statement does not endorse circular arguments.

Bolzano's position is easily misunderstood. When he later wrote, for example:

> The only reason why we are so certain that the rules *Barbara, Celarent*, etc., are valid is because they have been confirmed in thousands of arguments in which we have applied them. This also is the true reason why we are so confident, in mathematics, that factors in a different order give the same product, or that the sum of the angles in a triangle is equal to two right angles, or that the forces on a lever are in equilibrium when they stand in the inverse relation of their distances from the fulcrum, etc. (Bolzano, 1837, §315 [III.244]).

Coffa took him to be claiming that the grounds of general logical and mathematical claims lie in the particular instances, that they "derive from below, from the facts" (Coffa, 1992, 38). But Bolzano was thinking here only of the grounds of our conviction, or certainty. The question of *objective* grounds is an entirely different matter, and his remarks in the *Contributions* show that he already had a sophisticated view of the relations between the two. There, he observes that "our most vivid and clear judgements are obviously *derivative*. The proposition that a curved line is longer than the straight line between the same points is far clearer and obvious than many of those from which it must be laboriously derived" (Bolzano, 1810, II, §21, note). Often, he continues, we become convinced of the truth of an axiom precisely by noticing that it can be used to deduce consequences that are already recognised as true (1810, II, §21, note). The results we are most certain of, as Russell would later observe, lie somewhere in the middle: they are neither the most fundamental propositions of a science nor its remote consequences (Russell, 1973). The business of foundational research is to determine a set of axioms from which we may prove all results deemed certain and none that are deemed certainly false. Today, the point is easy to illustrate by analogy: when writing software, what we want to accomplish is often obvious enough. Much less obvious is how to get the job done, especially in machine language.

It would be difficult to exaggerate Bolzano's radicalism on the subject of axioms. For him, these are, quite literally, propositions that have no ground of their truth:

> With genuine axioms [*Grundsätze*], no ground is thought why the predicate belongs to the subject. For this ground would have to be another judgment. Now one might well counter that the ground of why

> the predicate belongs to the subject may lie in the subject and predicate themselves. But with a little reflection one will easily recognise that if this ground does not lie in one or several new judgments, the expression 'the ground lies in the subject or the predicate' just says: it's that way because that's the way it is, or the ground why this predicate belongs to this subject lies in the fact that this predicate belongs to this subject, i.e., it is grounded in itself, i.e., in other words, it has no ground (Bolzano 1977a I, §13).

Our ingrained habit of looking for reasons for truths reaches an impasse here—since axioms are true primitives, there simply is nothing prior to them in the relevant sense. Nor is there any point in looking outside the axiomatic system. In particular, anyone who thinks that an appeal to the essences of things or the constitution of the mind is just deluding himself:

> . . . people sometimes say [in such cases] that the ground lies in the absolute necessity of things, or else in the particular characteristics of our understanding—these are, I believe, empty ways of speaking, which in the end say no more than: "it's that way because . . . that's the way it is" (Bolzano 1810, appendix, §5, note).

Interestingly, Bolzano's rejection of the Cartesian understanding of the primitive elements of axiomatic systems has a early eighteenth-century precedent. Recall that Descartes had proposed his method for use not only for mathematics, but for science in general. His hopes for the universal application of his method had fared particularly poorly in physics, where Newton's *Principia* had relegated Descartes' apriorism to the status of an historical curiosity. There was resistance, to be sure. Cartesians complained that Newton's physics had no adequate foundation, because the primitive notion of gravitation, far from being self-evident, was perhaps even unintelligible. In the preface to the second edition of the *Principia*, Roger Cotes answered these critics as follows:

> But shall gravity be therefore called an occult cause, and thrown out of philosophy, because the cause of gravity is occult and not yet discovered? Those who affirm this, should be careful not to fall into an absurdity that may overturn the foundations of all philosophy. For causes usually proceed in a continued chain from those that are more compounded to those that are more simple; when we are arrived at the most simple cause we can go no farther. Therefore no mechanical account or explanation of the most simple cause is to be expected or given; for if it could be given, the cause would not be the most simple.

> These most simple causes will you then call occult, and reject them? Then you must reject those that immediately depend upon them, and those which depend upon these last, till philosophy is quite cleared and disencumbered of all causes.[4]

"Just so," one can imagine Bolzano saying, "and mathematics is no different."

Bolzano does not define the ground-consequence relation, but having claimed that it exists, he attempts to characterize it in part by giving an (incomplete) list of some simple and independent logical rules of inference which, he thinks, reflect objective relations of dependence (1810, II, §12). We may thus conclude that, for Bolzano in 1810, the grounding relation consists in a logically correct inference of a truth according to these rules from premises which are in themselves and necessarily the grounds of the conclusion. These rules of inference are so to speak the embryo of his logic of the ground-consequence (*Abfolge*) relation developed in the *Theory of Science* (Bolzano 1837, §§162 and 198-221).

There is just one simple independent syllogistic rule, *Barbara*:

S is M,
M is P,
———
S is P.

According to Bolzano, all other syllogistic forms either are not essentially different from *Barbara*, or are not simple.

Bolzano then introduces new rules of inference involving the conjunctions *et* and *cum*. He does not explain the meaning of the concepts designated by '*et*' and '*cum*' in the first issue of the *Contributions*, but does so in the unpublished second installment (Bolzano 1977a, §32 ff). According to what he says there, '*et*' represents *ideal* combination, which is possible between any two concepts, while '*cum*', by contrast, represents *real* combination, which is not always possible. The concepts "circle" and "square", for example, can be combined ideally (since one can think of a circle along with a square), but not really (since, as he then maintained, one cannot even form the concept of a circle which is square). One expedient that seems to work fairly well is to read '*et*' as 'along with' or 'as well as' and '$A\ cum\ B$' as 'A, which is B'. These, then, are the inference forms he thinks reflect genuine relations of grounds to consequence:

A is (or contains) B,
A is (or contains) C,
———
A is (or contains) [B *et* C].

[4] Isaac Newton, *Mathematical Principles of Natural Philosophy and his System of the World* tr. A. Motte, revised by F. Cajori (Berkeley and Los Angeles: University of California Press, 1966), Vol. I., p. xxvii.

A is (or contains) M,
B is (or contains) M.
―――――――――――――
[A *et* B] is (or contains) M

A is (or contains) M,
[A *cum* B] is possible, or A can contain B,
―――――――――――――
[A *cum* B] is (or contains) M.

On the other hand, perfectly valid inferences such as

[A *et* B] is (or contains) M
―――――――――――――
A is (or contains) M

are not instances of the consequence relation according to him, because the premise is not in itself the necessary ground of the two conclusions.

In attempting to elaborate his account of ground and consequence, Bolzano was following two hunches: first, that as we progress from grounds to consequences, we by and large progress from more simple to more complex truths; and second, that we also progress, by and large, from more general to more specific truths. We can see the influence of the former in the rules he endorses: in all four cases, the complexity (measured in terms of the complexity of subject and/or predicate) grows as we progress from the premises to the conclusion,[5] while in the rejected rule, the premise is more complex than the conclusion.

The second hunch makes itself felt in Bolzano's remarks on intermediate concepts in proofs, and in particular his endorsement of the Aristotelian command to avoid a transition to another genus, μεταβασις ἐις ἀλλο γενος (Bolzano, 1804, preface; Aristotle 1995, I, 7).

Bolzano attempted to combine the two ideas by stating additional conditions that must be satisfied by the intermediate concepts in the forms of inference enumerated above (Bolzano 1810, II, §29). These attempts were not obviously successful, and Bolzano would later record his second thoughts in one of his notebooks (see Centrone 2011 for a more detailed discussion).

Still, there can be no doubt about the heuristic force of the motivating ideas. For example, he maintained that proofs in the more general science of analysis should not make use of principles drawn from the more special science of geometry. For a geometrical proof of a theorem of analysis will always confront the following dilemma: either all of its premises hold not only for spatial quantities, but instead for continuous quantities of all kinds, or else (provided it does not contain any

[5]To make this perfectly clear, the last of the four inferences would be better expressed as: A is M, A can be B; hence [A *cum* B] is M.

idle premises) it will make essential use of premises that, while valid for spatial quantities, are not valid for continuous quantities in general. In the former case, the geometrical concepts may and should be omitted, in the latter, the proof is inevitably invalid. Either way, the geometrical considerations have no place in a correct proof. He points out by way of example that Lagrange deduced one of his key analytical theorems from a property of continuous curves, thus from a geometrical consideration, while it is precisely the continuity of the function that guarantees the continuity of the corresponding curve. Thus his proof, along with many similar ones, is simply circular (Bolzano 1810, II, §29, note; cf. Rusnock 1997b, p. 68-70).

Mathesis universalis

After the publication of the first issue of the *Contributions*, Bolzano sketched the second issue under the title *Mathesis universalis*, but did not finish it (Bolzano 1977a). In this manuscript, he tries to determine the frontiers of this discipline and further develops his criticisms of Kant. Let us return to his definition of mathematics as the science of laws governing all things without exception: what are the laws "to which things must conform in their existence"? The question falls within the province of ontology which, according to Wolff, is supposed "to prove the attributes of all beings (*entia*) either absolutely or under a certain condition" (Wolff 1730, Prolegomena, §8 [p. 5]). Bolzano finds two such laws, but before quoting them, we shall examine with him the principles that are claimed to be universal.

First, he excludes the logical principles of non-contradiction, excluded middle, and identity, because they are analytic and even identical judgments. In an important observation, Bolzano explains the difference between identity and equality: when one says that a thing is equal to itself,

> . . . it would be actually more correct to say that every thing is identical to itself. When we have two ideas (representations) of the same thing and do not know that they are ideas of the same thing, we suppose first that there are two things, called for example A and B; later we find that A and B are identical (1977a, I, §13).

Frege would later illustrate Bolzano's observation with the example of the morning star and the evening star.

Let us now consider genuine ontological principles. The principle of universal determination is true of every object, but Bolzano reduces it to the principle of excluded middle: "Everything is A or non-A" (1977a, I, §8).

The different versions of the principle of sufficient reason are all inadequate (1977a, I, §11). Kant limits its use to phenomena, but, as Bolzano remarks, he tacitly assumes it in claiming things in themselves to be the ground of phenomena. This principle is for Bolzano simple, ungrounded and its validity is limited. Objective principles (axioms) in general do not depend on further reasons; moreover, free actions and even the existence of God have no ground. From his analysis Bolzano draws a radical conclusion: *there is no principle of sufficient reason having universal validity*, because there is no universal procedure to decide if any given thing has a sufficient reason (1977a, I, §15).

Other principles and maxims are objects of Bolzano's criticism: the principle of similitude and the scholastic maxim of the immutability of things (1977a, I, §16-17). He then returns to Kant, reviews the table of the principles of understanding, which he claims do not belong to universal mathematics because their domain of validity is limited (I, §§21-23). Let us notice his critical remark on the Kantian concept of possibility. Bolzano agrees with Kant that possibility (and other modal notions) does not contain a new determination of the concept to which it is appended. It has another role, namely to express the *formation* of a concept. To say that a right-angled triangle is possible is to say that a triangle *can be* right-angled. Thus, the concept of possibility is not a predicate, but a copula. The concept of possibility turns into an *attempt* to construct a complex concept, an attempt that can produce an authentic concept, but that also can fail and result in a bare accumulation of signs having no sense. In this text, even Wolff's example of a bi-angle (a figure enclosed by two straight lines) does not correspond to a concept; in the *Theory of Science* and later works, by contrast, such expressions do designate (objectless) concepts, as opposed to bare accumulations of signs like x/: (1977a, I, §25; Bolzano 1837, §67).

What, then, are the principles of universal mathematics or, in Husserl's terms, of formal ontology? There are two: (Bolzano 1977a, I, §3).

1. the law of the possibility of "*thinking-together*" (*Zusammendenkbarkeit*) several things, according to which any thing can be joined in thought to any other thing; and

2. the law of *relation*, stating that any thing bears a certain relation to every other thing (Bolzano says almost nothing about this second law).

At this stage, Bolzano thinks that these laws only govern our ideas, and are not valid for the things themselves and their existence. Later, he would change his mind on this point, holding that collections subsist even if no one thinks of uniting their elements (Bolzano 1851, §§3, 14; Bolzano 1975b, III, §6).

When stated as follows: "all things can be ideally united", the first law is very close to Cantor's much later and far more famous definition of a set: "By set, we understand any union (*Zusammensetzung*) M of definite and well distinguished objects m of our intuition or of our thought conceived as a whole" (Cantor, 1966, 282). Bolzano calls the results of such union a *whole* (*ein Ganzes*) or a *system* and sometimes also a *sum* (in his later works, Bolzano prefers the term *collection*, [*Inbegriff*]) and the things united in a whole its parts (*Teile*).[6] No homogeneity of parts is necessary to form a whole; thus, to cite some extreme cases, it is possible to unite a candle (a real thing) and a syllogism (a sequence of propositions in themselves) in a whole (Bolzano 2004b, 142, 160), and the *Paradoxes of the infinite* (1851, §3) gives as examples the collection formed by a rose and the concept of rose, and that containing the name of Socrates and a definite description of him. It is important, too, to keep in mind that a whole can have infinitely many parts—for example, the whole consisting of the points of a line.

Bolzano's wholes cannot be immediately identified with modern sets for a number of reasons: to begin with, collections are complex by definition, ruling out not only the empty set but also singletons. In addition, wholes, unlike sets, are generally endowed with a certain structure. To determine a whole thus requires us to determine both its parts *and* the manner of their combination.

Already in his *Dissertation on the combinatorial art* (1666), Leibniz postulated that it is possible "to take together simultaneously (*simul sumere*) any objects and to suppose that they form a whole", but the idea that the notion of a whole or system could occupy a central place in mathematics disappeared from the horizon of mathematicians. Bolzano's seminal idea brings it back. Throughout his scientific career, Bolzano will not only use it, but will consider it to be *the fundamental concept* of mathematics: quantities are arithmetical wholes determined by numbers, numbers are discrete quantities, which is to say, sets [*Mengen*] (Bolzano 1977a, III, §15). Like Euclid, Bolzano draws a sharp line between numbers, i.e., natural numbers, and quantities [*Grössen*], which are our positive, negative, rational, and irrational numbers. The geometrical objects Bolzano calls spatial objects [*Raumdinge*] are also collections. "A spatial object is in general any system (any collection) of points (which may form a finite or an infinite set)" (1817b, §11); lines, surfaces, solids are such systems of points.

In the *Theory of Functions*, Bolzano uses the concept of collection and set in exactly the same way as in the works of Weierstrass and of his school (Bolzano 2000). Even if the title *mathesis universalis* does not appear in the *Theory of quantity*, the idea still seems to animate his theory of collections. In its most accomplished form in the *Theory of Quantity*, it presents two fundamental principles of Cantorian set theory: an extensionality principle governing sets (*Mengen*) and a principle of comprehension (Bolzano 1975b, III, §89; 1851, §14). The most im-

[6]For further discussion of Bolzano's theory of collections see Krickel 1995, Behboud 1977, Simons 1977, Vopěnka, 1997 and 1998.

portant insight of the *Paradoxes of the infinite*, a book carefully studied both by Cantor and by Dedekind, states the characteristic property of infinite sets, their reflexivity, i.e., the existence of a 1-1 correspondence between a set and one of its proper subsets (1851, §20). Dedekind (1965, §5, def. 64 and th. 66) would later use this property to define the concept of an infinite set.

Conclusion

Although still indebted to Kant and adopting some of his important distinctions, in the *Contributions* Bolzano is already moving in the opposite direction, stressing the objective basis of human knowledge and the logical structure of science. In this work, which should have had the kind of influence on nineteenth-century philosophy that Wittgenstein's *Tractatus* had on that of the twentieth, Bolzano puts forward his main themes and concepts: the objective connection between truths based on the concept of grounding, the two kinds of proofs: certifications and objective groundings, the idea of contextual definition, and the theory of collections as the most basic part of mathematics. In addition to these positive contributions, he offers a refutation of Kant's philosophy, especially of his philosophy of mathematics based on the contradictory concept of pure intuition. In opposition to Kant, Bolzano founded his philosophy on logic, whose central idea is that of objective proof.

This conception soon showed its fruitfulness in his works on the infinitesimal calculus: in the *Purely analytical proof* and in the *Theory of Functions*. Both introduced new rigor, new arithmetico-analytical methods and new concepts, preparing the way for the school of Weierstrass. The recognition of his logic came only in the twentieth century when Tarski and Carnap elaborated logical semantics. The seeds of these developments lie in the small booklet published in Prague two hundred years ago.

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