Philosophy of mathematics: Bolzano's responses to Kant and Lagrange

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In a late essay, Bolzano describes the philosophy of mathematics as an activity aimed at discovering the objective grounds of propositions which we already know with the greatest certainty and evidence.[1] For him, philosophy of mathematics was simply what we would now call foundational research in the broadest sense—that is, it was not just a matter of "ultimate" foundations (e. g. set theory, logic, or the like), but also of the foundations of particular mathematical theories (e. g. geometry, the calculus, combinatorics, . . .). Bolzano was certainly committed to dealing with questions of ultimate foundations, with developing a unified system of mathematics from first principles—his detailed investigations of set theory and logic bear ample witness to this. He also understood, however, that foundational inquiries could be, at least provisionally, local. One could, as he explained in the *Contributions to a better-founded presentation of mathematics* of 1810, assume certain propositions as locally primitive, deferring until a later date their proof from more basic principles.[2] No sharp line can be drawn to separate such local questions from those of ultimate foundations. Searching for underlying principles, in whatever domain and at whatever level, was an activity he quite plausibly and in line with tradition regarded as philosophical.

The results of Bolzanian philosophical inquiries in mathematics are very much like (interpreted) Hilbertian formal systems: specifications of primitive concepts, definitions of other important concepts in terms of these, a specification of axioms, and proofs (meeting certain strict criteria and illustrating generally useful techniques) of a selection of important theorems. The specification of such a system is

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[1] Bernard Bolzano, *Was ist Philosophie?* (Wien, 1849), 23.

[2] Bernard Bolzano, *Beyträge zu einer begründeteren Darstellung der Mathematik* (Prague, 1810), II, §11. (Hereafter: Beyträge).

precisely what he has in mind when he speaks of presenting the objective grounds of a proposition. Thus, for example, his first published work in analysis[3] showed that and why Newton's well-known binomial series:

$$1 + nx + n \cdot \frac{(n-1)}{2}x^2 + \cdots + \frac{n!}{r!(n-r)!}x^r + \cdots$$

converged to the value $(1+x)^n$, for $|x| < 1$ by embedding this particular result in a general theory of power series. His proof of the intermediate value theorem,[4] similarly, proceeded by setting out enough of the formal structure of real function theory to support the result. These and similar foundational inquiries were Bolzano's favorite kind of mathematics—the kind (as he described it in his autobiography) "which is at the same time philosophy."[5]

Bolzano's understanding of the philosophy of mathematics had a certain currency in his day—one recalls, for instance, the learned discussions of *la métaphysique du calcul infinitésimal* carried on through the late eighteenth and early nineteenth century, or Gauss's foundational sketch "*Zur Metaphysik der Mathematik*." It is, evidently, not one which has survived. Almost no one today would say that the foundational work of Cauchy, Weierstrass, or Lebesgue belonged to the philosophy of mathematics (though it might be thought to illustrate or illuminate various important philosophical questions). Foundational inquiries, apart from those which concern what might be called the "initial" fragments of mathematics, are considered mathematical but rarely if ever philosophical. And if Bolzano is discussed as a philosopher of mathematics today, it is not so much for the sake of his foundational research (his philosophy of mathematics) as for the methodological investigations which surround and inform it—investigations which for Bolzano do not belong to the philosophy of mathematics at all, but rather to the more general theory of science, or logic.[6]

One can thus speak of Bolzano's philosophy of mathematics in two distinct senses: as logic in his generous sense of the word and as foundations. In this essay, I would like to do just that. In particular, in order to illustrate the character of his work in these two areas, I will consider Bolzano's responses to two of his most influential precursors, Kant and Lagrange. It is striking that despite the vast differences in the subjects treated, Bolzano's method is the same in both cases. He

[3] Bernard Bolzano, *Der binomische Lehrsatz. . .* (Prag, 1816).

[4] Bernard Bolzano, *Rein analytischer Beweis . . .* (Prag, 1817).

[5] Bernard Bolzano, *Lebensbeschreibung des Dr. B. Bolzano* (Sulzbach, 1836), 19.

[6] Bernard Bolzano, *Beyträge*, II, §1: ". . . the mathematical method is at bottom nothing but . . . Logic, and thus does not belong to mathematics at all." It is important to note that for Bolzano logic included not only the topics generally called logical today, but also (among other things) epistemology and heuristics.

begins, namely, with conceptual analysis: an attempt to determine as precisely as possible what the concepts and propositions employed by these authors mean, or might reasonably be taken to mean. Only then does he proceed to an assessment of their claims, to see what can justifiably be kept and what must be rejected. In the case of Kant, there was in Bolzano's view little worth retaining. Although he kept some of Kant's terminology, and attempted to assign a reasonable meaning to some of his concepts and statements, Bolzano's position was overwhelmingly one of opposition. With Lagrange, things were quite otherwise: although his criticisms of Lagrange's development of functional analysis were just as thorough and deep as his criticisms of Kant's theory of mathematical knowledge, what he retained from Lagrange was far more substantial.

Bolzano and Kant

To one unfamiliar with the ways of nineteenth-century German-language philosophy, Bolzano's concern with Kant's account of mathematical knowledge should seem eccentric. Kant, after all, had very little training in and a limited knowledge of mathematics[7] while Bolzano knew the literature well and made contributions of the first rank to the field. Again, there would seem to have been relatively little chance of Kant's views having any influence for better or worse on the development of mathematics at that time. For the indisputable centre of advanced research then was Paris, and there, in addition to linguistic and cultural barriers to German thought,[8] mathematics had been effectively isolated from academic philosophy with the founding of the *Grandes Écoles*. (Gauss, a rare exception to the relative backwardness of German mathematics, was no great friend of Kant's opinions on the subject.[9])

[7]This has been documented at length, with, however, little apparent effect on general philosophical opinion, by Erich Adickes, *Kant als Naturforscher*, (Berlin: De Gruyter, 1924).

[8]Instructive, in this regard, is the report of Destutt de Tracy on Kant's metaphysics: "De la métaphysique de Kant…," par le citoyen Destutt de Tracy, lu le 7 floréal, an 10 (27 April 1802); printed in *Mémoires de l'Institut National des Sciences et Arts pour l'an IV de la République. Sciences Morales et politiques*, Tome 4e (Paris, Vendémiaire, an XI) reprinted in Antoine Louis Claude Destutt de Tracy, *Mémoire sur la faculté de penser, De la Métaphysique de Kant, et autres textes* (Fayard, 1992).

[9]See, e.g. his letter to Schumacher of 1 November 1844, (Werke, 12.62-63): "That you think a philosopher *ex professo* incapable of confused concepts and definitions almost astounds me. Nowhere are such things so much at home as with philosophers who are not mathematicians, and Wolff was no mathematician, even if he did turn out cheap compendia. Just take a glance at some contemporary philosophers, at Schelling, Hegel, Nees von Esenbeck, and the like, don't their definitions make your hair stand on end? …But even with Kant it is often not much better. His distinction between analytic and synthetic propositions [i.e. a cornerstone of Kant's account of mathematics] is in my opinion one which either amounts to a triviality or is false."

Bolzano's concern was nevertheless a reasonable one. Kant's views, no matter how poorly they accorded with or irrelevant they were to mathematical practice, had considerable currency among philosophers.[10] And since mathematics and its methods had always occupied a central place in philosophical reflections on science, Kant's views on the topic had to be taken seriously.[11] But it was not only the philosophers who followed Kant: some academic mathematicians in Germany, like Kant's colleague Schultz, writers of textbooks, manuals, and so on, were also visibly under the influence of Kant's doctrines. The more misleading the tools of learning produced by them were, the more time was likely to be lost correcting misapprehensions when, as was inevitable, German-language mathematics began to catch up. For one as acutely sensitive to the waste of time as Bolzano, Kant's doctrines needed urgently to be refuted in the most public way.

Kant's opinions on mathematics are familiar to most, so only the briefest summary is called for here. At their centre are the claims that from concepts alone only analytic judgments (in Kant's narrow sense of the term) can be derived; and that most of the significant content of mathematics consists in synthetic judgments *a priori*. Being synthetic, such judgments cannot be derived by analysis which, for Kant, means they cannot be derived by purely conceptual means.[12] They must, accordingly, be established some other way: and since the only other sorts of cognitions he admitted were intuitions,[13] these judgments must, he thought, be somehow mediated by intuitions. Now in the case of mathematics, we are able to generate objects (or rather, representations of objects, or intuitions) for mathematical concepts, either in imagination or on paper, blackboards, etc. ; and, by considering the properties of these intuitions, we are able to derive judgments which concepts alone could not yield. Thus to prove a theorem about all triangles, we procure an intuition of a single triangle, and after performing various constructions observe that it has the stated property. That this procedure does not simply yield particular judgments is due to the circumstance that the "single figure which we draw

[10]In the *Theory of Science*, for example, Bolzano wrote that Kant's theory of time and space had been accepted by almost all German philosophers. *Wissenschaftslehre* (Sulzbach, 1837) §79, Note.

[11]Another reason for Bolzano's concern with Kant stemmed from his opinion that mathematical training of a certain kind was an almost indispensable preparation for any serious philosophical research. Since Kant had stated on many occasions that the methods of mathematics and philosophy had nothing in common, he was, at least on the face of things, strongly opposed to Bolzano on this point.

[12]Cf. Immanuel Kant, *Der einzige mögliche Beweisgrund zu einer Demonstration des Daseins Gottes*, in *Kant's gesammelte Schriften* (Berlin: G. Reimer, 1910-) t. 2, p. 156, tr. Walford and Meerbote : "…the only way in which it is possible to derive a consequence from a concept of the possible is by logical analysis."

[13]See, e.g., Immanuel Kant, *Critique of Pure Reason*, tr. Norman Kemp Smith (New York: St. Martin's Press, 1965), A 320.

serves to express the concept without impairing its universality," in that we "abstract from the many determinations (for instance, the magnitude of the sides and of the angles) which are quite indifferent, as not altering the concept 'triangle'."[14] The perspicuous instance, correctly understood, establishes the general result.

Against this, Bolzano (the first, by the way, to construct a fractal[15]) objected first of all that we cannot readily conjure up images for all mathematical concepts, and yet this does not prevent us from dealing successfully with them. To the proposition that each straight line may be extended *ad infinitum*, for example, there corresponds no intuition, for we have no intuition of infinitely long lines (Bolzano might have added, but didn't: still less of two infinitely long lines, in order to verify, for instance, the axiom of parallels). "In stereometry we often deal with intricate spatial objects which even the most vivid imagination cannot clearly represent; nonetheless we continue to calculate with our concepts, and arrive at truth."[16] Outside of geometry, where Kant appealed to so-called symbolic constructions, the claim is still less plausible.[17]

More telling still is Bolzano's logical criticism. To begin with, he notes that two sorts of proofs should be distinguished: those which merely aim at convincing that a proposition is true, and those which give what he calls its objective ground (this, in Aristotle's words, is the distinction between proofs that show *that* and those which also show *why* a proposition is true). It is the second sort of proof which interests Bolzano above all: such proofs must meet some rather strong logical criteria, and, if they do, can not only secure conviction in the truth of the result but, more importantly, also help us to discover new, previously unattainable truths. The goal is not so much certainty as insight. Thus in his first publication, Bolzano countered Kant's portrait of Thales[18] with his own:

> It might have seemed superfluous when Thales (or whoever else it was who discovered the first geometrical proof) gave himself a good deal of trouble to demonstrate that the base angles of an isosceles triangle are equal, when this is obvious to the most common human understanding; but he did not doubt in the slightest that it was so, but rather only wanted to know why the understanding made this necessary claim. And behold! in drawing out the elements of a hidden inference and becoming distinctly conscious of them, he also obtained

[14] Immanuel Kant, *Critique of Pure Reason*, A 713 f.

[15] Bernard Bolzano, *Functionenlehre*, ed. K. Rychlik (Prag, 1930), II, §75

[16] Bernard Bolzano, *Beyträge*, Anhang, §9.

[17] See Bernard Bolzano, *Von der mathematischen Lehrart*, ed. Jan Berg (Stuttgart-Bad Cannstatt: Fromann-Holzboog, 1981), §11, note 1, for a spirited polemic on this topic (here directed against Fries).

[18] Immanuel Kant, *Critique of Pure Reason*, B xi-xii.

> the key to new truths which are no longer obvious to the common understanding.[19]

One important minimal criterion for an objective proof is that it does not include any inferences from the special to the general[20]—which, on the face of things, is all that Kant's mediations of judgments by intuitions are. In using intuitions in the manner suggested by Kant, Bolzano remarks, one tacitly relies on certain purely conceptual truths:

> If, for example, we recognize the truth that every straight line can be extended from the consideration of a determinate straight line, one would say that we had arrived at this cognition by means of an intuition. Yet this much is certain: we can only infer such a general truth from that which the appearance of only a single line teaches us insofar as we assume the other truth that all straight lines are similar to one another, and thus that every inner attribute which is representable by means of concepts found in one must also be found to reside in every other. This proposition, however, no intuition can teach us.[21]

Once the similarity of all straight lines in Bolzano's sense is conceded—something which is plausibly enough claimed to be a necessary part of such a proof—the appeal to intuition crumbles. For either this similarity is established by the instances—and thus the appeal to a single intuition requires us to examine an infinity of others—or else the generality must already be present with the concept.[22]

This, of course, was not all there was to Kant's account of mathematical knowledge, which was hedged round with the usual collection of qualifications scattered here and there. The universality expressed by intuitions can be attributed to their being determined by certain "universal conditions of construction" (A 714), and indeed we are even told that it is not intuitions but schemata which truly play the

[19] Bernard Bolzano, *Betrachtungen über einige Gegenstände der Elementargeometrie* (Prague, 1804), Vorrede.

[20] See Bolzano's discussion in the *Beyträge*, II, §29.

[21] Bernard Bolzano, *Von der mathematischen Lehrart*, §14, Note 2.

[22] Cf. *Beyträge*, Anhang, §7: "Kant appears to have intended to say: 'When I connect the general concept of, for instance, a point, or of a direction or a distance with an intuition, i.e. represent to myself an individual point, or an individual direction or distance, then I find that this or that predicate belongs to these individual objects, and feel at the same time that this is the case for all other objects which fall under these concepts.' If this be the opinion of Kant and his followers, then I ask: how is it that with these intuitions I have the feeling that what I observe in them also belongs to all the others? Through that which is single and individual, or through that which is general in the object? Clearly only through the latter—i.e. through the concept, not through the intuition."

mediating role.[23] By patching together enough partial retractions, it is possible to make Kant's views seem much more reasonable. But in so doing they quickly lose their distinctiveness. If, for example, the "universal conditions of construction" are made explicit—that is, their conceptual structure distinctly set out, it is difficult to see what role would remain for the intuitions. The schemata of geometry, for their part, are not intuitive (as Kant himself conceded[24]) but rather a certain restricted kind of concept, which Bolzano plausibly enough identifies with the so-called genetic definitions of geometry.[25] This being the case, Bolzano observed, "...if one recognizes the truth of a synthetic proposition from the consideration of the schema of its subject-concept, one recognizes its truth from a mere concept."[26]

Bolzano did not stop with these few words of refutation, but dealt at length with many of the further explications of Kant and his followers. In general, he shows an admirable patience and respect for their opinions, seeking always to find something of value in them; but to those of their doctrines he judged false he gave no quarter.[27] To discuss these matters here would require far too much time and cast little light; instead, I will attempt to sketch Bolzano's positive response to Kant, his alternative versions, in particular, of some of the central concepts of Kant's epistemology and logic. Here we find a refutation which goes beyond the obvious or surface defects of Kant's account and opposes it on a more fundamental level.

I begin with what is perhaps the most striking difference between the two, namely, their understanding of the objects of logic. Kant's logic and epistemology are firmly rooted in the world of ideas, that is, representations of things (*Erkenntnisse* in his terminology), typically individuals and their properties. Traditionally, ideas do duty for objects in the mind, and were frequently conceived as fully formed copies of the things represented. Kant altered this picture somewhat by locating the source of many of the features of these objects within the cogitating subject, but his cognitions are nevertheless very much thing-like. An idea of an individual must, for him, not only have only one object, it must also reflect all the determinations of the object. So, for example, supposing there to be only one white crow in the world, the concept "white crow" would still not be a representation of

[23] Immanuel Kant, *Critique of Pure Reason*, B180: "Indeed it is schemata, not images of objects, which underlie our pure sensible concepts. No image could ever be adequate to the concept of a triangle in general. It would never attain that universality of the concept which renders it valid of all triangle, whether right-angled, obtuse-angled, or acute-angled .... The schema of the triangle can exist nowhere but in thought. It is a rule of synthesis of the imagination, in respect to pure figures in space."

[24] Immanuel Kant, *Critique of Pure Reason*, A141/B180.

[25] Bernard Bolzano, *Wissenschaftslehre*, §305 (3.183); cf. Kant, *Logik* ed. Jäsche, in *Kant's Werke* (Berlin: De Gruyter, 1923) t. 9, §106, where Kant states that all mathematical definitions are genetic.

[26] *Ibid.*

[27] See, for example, his dogged pursuit of Schultz in §79 of the *Wissenschaftslehre*.

an individual in his usual sense. That is, by its form it is a general representation, and applies to a single object only by accident—for we would not know, for example, whether the crow was male or female, whether its feathers were symmetrically disposed, etc. Only when we know all this can we say that we have represented an individual. Since only an infinite list of characteristics, or concepts, could do this, representations of individuals cannot be conceptual—they are heterogeneous, namely, intuitions.[28] Intuitions, with their image character, can reflect the infinite (or indefinite) complexity of individuals. We can always find more detail in them, which is why, as Kant notes, a telescope is handy for making some of our intuitions more distinct.[29] Conceptual analysis, by contrast, always ends after a finite number of steps, when we arrive at simple component concepts.

Bolzano also speaks of representations, but for him ideas are not stand-ins for things, nor do they resemble things; rather they are much more like linguistic entities, or signifiers. According to his official definition, a representation is a part of a proposition which is not itself a complete proposition.[30] Propositions, for their part, are characterized[31] as a kind of abstract or mathematical object, namely, structured truth-bearers without real existence. A representation does not have to take the place of an object, reflecting all its determinations; rather, having objects is a primitive semantic relation: a representation simply has (or lacks) objects, and may have, according to circumstances, zero, one, several, or even infinitely many of them. Among the noteworthy representations with zero objects are such ideas as "have", "and", "not" "there exist"—ideas which do not count as cognitions in Kant's account, since (unlike "Caesar" or "dog") they do not present any object. Nor does the circumstance that an idea has only one object mean it has to be an intuition: the ideas "God", "universe", "even prime number", etc. are purely conceptual, yet singular. Images, although they can be objects of representations, are not themselves representations for Bolzano, nor do representations resemble them. Thus where Kant thought that an intuition of a house must contain representations of doors, windows, etc. as parts,[32] this is not the case for Bolzano. He too, as we

[28]I. Kant, *Logik* ed. Jäsche, §15, tr. Hartmann and Scwartz : "By continued logical abstraction originate ever higher concepts, just as on the other hand ever lower concepts originate by continued logical determination. The greatest possible abstraction gives the highest or most abstract concept—that from which thinking can remove no further determination. The highest complete determination would give the all-sided determination of a concept, i.e. one to which thinking can add no further determination." He adds in a note: "Since only single things or individuals are of an all-sided determination, there can be cognitions of an all-sided determination only as intuitions, not, however, as concepts; in respect of the latter, logical determination can never be considered complete."

[29]Immanuel Kant, *Logik* ed. Jäsche, Introduction, V.

[30]Bernard Bolzano, *Wissenschaftslehre*, §48.

[31]Bernard Bolzano, *Wissenschaftslehre*, §19.

[32]Immanuel Kant, *Logik* ed. Jäsche, Introduction, V.

shall see, speaks of intuitions, but his, like Leibniz's monads, have no windows.

Neither, notoriously, are representations as such mental entities. Although Bolzano considers subjective, or thought representations at length, his central logical concepts are defined first for objective propositions and representations, or "representations in themselves". Although his remarks on this topic have proved quite difficult for many to digest,[33] Bolzano saw clearly that logic—the theory of representation, inference, etc. —could and should be investigated on this level, without reference to the mind or its faculties. Thus where Kant could not simply say that a predicate B applied to a subject A because every object which stands under A also stands under B—asking instead why the mind saw fit to attach B to A—Bolzano has truth as a semantic relation defined independently of the mind. Kant's distinction between analytic and synthetic judgments was indeed an important one in Bolzano's view, but not at all for the reasons Kant had adduced; rather, it was a valuable observation because it made it clear that it is not the case that a representation of an object must contain a representation of each attribute of the object as a part.[34] A list of all the attributes of triangles is one thing, the list of the parts of a phrase like "plane figure bounded by three straight lines" quite another; and, for Bolzano, as for Kant, concepts are structured like the latter rather than the former. Indeed, if the view Kant sought to rebut were true, a concept like "triangle" would have to have infinitely many parts, since triangles have infinitely many properties—a view, by the way, that the authors of the Port Royal Logic had not shied away from. By insisting on the importance of the distinction between analytic and synthetic judgments, Bolzano thought, Kant had brought a measure of sanity to the theory of ideas on this point.

Bolzano, for reasons just noted, was not as troubled as Kant over the question why the mind attached certain predicates to certain subjects, but nevertheless offered a good deal more of substance on the subject. As soon as one admits simple concepts, as he argued in an essay of 1810, one will also have to admit primitive predications (i. e. axioms) which are synthetic in Kant's sense. These, along with definitions and rules of inference, form the basis for the entire conceptual and deductive structure of mathematics. But the fact that such propositions are logically primitive does not mean that they must also be epistemologically primitive—for instance, self-evident, or otherwise independently justifiable. As one with experience constructing axiomatic systems, Bolzano knew that first principles were quite often less obvious than their consequences, and might even appear false as long as one had not grasped their role in supporting these consequences.[35] One justifies

[33]Much of his correspondence with Exner, for instance, turns on this one point.

[34]Bernard Bolzano, *Wissenschaftslehre*, §65, 8c (1.288 f).

[35]Bernard Bolzano, *Beyträge*, II, §21.

their status as axioms—according to the account of 1810—first by establishing that they are unprovable, and second by showing that they do in fact support exactly the results one accepts as true.[36] This is Bolzano's version of Kant's deduction, or justification of principles; it is important to note that his, unlike Kant's, is both fallible and defeasible.

The fundamental difference between Kant and Bolzano on the nature of representations leads fairly directly to another concerning the capacities of purely conceptual reasoning. In the Kantian dispensation, concepts by themselves can yield only analytic judgments, those of the form "A, which is B, is B".[37] Bolzano's understanding of things is quite different. For him, to ask what can be established by means of concepts is more like asking what can be established with words or similar systems of symbols. And here, Kant's account—tied as it is to the traditional assumption that all ideas are either of individuals or their properties—falls far short of recognizing the resources at our disposal. Bolzano's alternative—foreshadowed by Leibniz's level-headed remarks on formal arguments in the *New Essays*[38]—puts these to work, using the technique of variation to set out incomparably richer concepts of compatibility, deducibility, equivalence, validity, analyticity, and probability.[39]

The concept of analyticity, for example, is defined as follows. A proposition is said to be analytic relative to a specification of certain parts as variable iff every allowable substitution of other parts for those considered variable leaves the truth value unchanged. Thus for example, the proposition expressed by the sentence:

> If Caius is older than Brutus, then Brutus is younger than Caius.

is analytic relative to "Caius" and "Brutus", since every appropriate substitution of other representations for these results in a true proposition. Consequence, or deducibility, he defines in a way reminiscent of Tarski. That is, a proposition Z is said to be a consequence of other propositions A, B, C, . . . relative to designated

[36] *Ibid.*; see also the discussion of J. Laz, *Bolzano critique de Kant* (Paris: Vrin, 1993).

[37] Kant's reasons for thinking this can be sketched as follows. The normal case—which is what his statement covers—is the universal affirmative judgment "All A are B". Concepts are conjunctions of characteristics. Thus a concept by itself will yield nothing but this list of characteristics. A synthetic judgment "A is C" (where C does not occur as a characteristic in the concept A) thus goes beyond the concept. To establish it, Kant thinks, we must have representations of one or more objects which fall under A and which have the property represented by C (the more A's represented which are found to be C, the greater confidence we can have in the judgment; in the case of *a priori* intuitions, however, a single one is believed to secure the result.)—note, by the way, that it is the *representations*, i.e. the intuitions, which are said to have the property, not the *objects* represented. Else the assertion is, as he puts it, empty.

[38] See especially Book IV.

[39] Bernard Bolzano, *Wissenschaftslehre*, §§147 ff.

occurrences of certain variable parts i, j, k, ... iff every allowable substitution of other parts $i', j', k', \ldots$ which makes all of A, B, C, ... true also makes Z true. Thus for example

$$5 > 3$$

is a consequence of

$$5 > 4$$

and

$$4 > 3$$

relative to the specification of all occurrences of "3", "4" and "5" as variable parts, for any substitutions for a, b, c in the propositional forms

$$a > b, b > c, a > c$$

which makes the first two true also makes the last one true. In conjunction with his careful definitions of mathematical concepts such as function, continuity, differentiability, etc., these logical notions show themselves to be well adapted to the practice of mathematics; and Bolzano's successes in mathematics show not only that Kant's intuitions were dispensable, but more significantly that appeals to intuition often held back the emergence of valuable new results.

Finally, a word about intuitions and their role in Bolzano's theory of science.[40] Bolzano clearly had no place for or need of Kantian intuitions in his logic or mathematics. Nevertheless, he thought that with the distinction between intuition and concept, as with the distinction between analytic and synthetic judgments, Kant had picked up the scent of something important, no matter how unsuitable his grasp or exposition of it.

Kant clearly thought of intuitions as image-like[41], but he also spoke of them in several other ways: as representations of individuals, as representations in immediate relation to their objects, and as representations which can occur as the subjects of judgments but never as predicates. Bolzano—who unlike Kant could never fathom how an image could find its way into a proposition—limits himself to these suggestions. The condition that a representation have only one object is insufficient by itself to make it an intuition: the representations "even prime number", "centre of mass of Jupiter"', "the universe" each have only one object, but are surely conceptual by Kant's reckoning. We fall back, then, on the second and third

[40]For a detailed study of this question, see Rolf George, "Intuitions, the theories of Kant and Bolzano," pp. 319-354 in M. Siebel and M. Textor ed. *Semantik und Ontologie* (Frankfurt: Ontos. 2004).

[41]See, for example, *Critique of Pure Reason*, A469/B497, A525/B553; also in the dispute with Eberhard, Akademie edition, 8.201, 8.205.

conditions. Let us begin with the third: intuitions can occur as subjects in judgements but ever as predicates. Here are some examples of intuitions cited by Kant (admittedly an odd lot): "Caesar", "The Sun", "Bucephalus", "Rome", "Time", "Space". Apart from the last two, which might stir some controversy, we can see why Kant might say that intuitions can only occur in subject position: for these are proper names, which do behave just like that in sentences. But proper names like "The sun" must surely give us pause: for this might easily be thought to be a description, not really different from "the third street past Main", which Kant would consider a concept rather than an intuition. To be a true proper name, Bolzano seems to have thought, a representation of a single individual would have to be not just a representation designated in German by a proper name, but itself what Russell would later call a logically proper name: that is, it would have to be simple, having no parts, no internal complexity. Thus Bolzano's definition of intuitions as representations which are simple and have only one object.

Like Russell, Bolzano thinks that we can only have such representations of objects we are directly acquainted with, thus making contact, if obliquely, with Kant's third requirement: immediacy. Like Russell, too, Bolzano thinks that we are not directly acquainted with those things usually designated by proper names: people, places, institutions, etc. What, then, are the objects of intuitions? In short: something like mental states. Intuitions serve as labels which allow us to refer to these states. Being simple, they function rather like demonstratives.[42]

Intuitions are thought by Bolzano to correspond to an important natural kind of subjective representations, those, namely, which lie at the interface between the conceptual, propositional functioning of the mind and the (pre-propositional) changes in us caused either by outer objects or our own mental activity.[43] Such subjective intuitions are, in his view, incommunicable and unrepeatable. If we had to discuss them, we would have to fall back on expressions with indexicals, like "this, which occurs in me just now or "This, which I just now see or hear or feel."[44] Underlying all of these, however, as a precondition of their being thought, is the bare idea which we designate, however inadequately, by "this", an idea which Bolzano claims to be simple.[45] In all cases, our subjective intuitions have objects

[42]The closest thing in Kant's scheme would be *sensations*—representations which, as he writes, "relate solely to the subject as the modification of its state." (A 320) But Bolzano's intuitions are not themselves sensations; rather, something like Kant's sensations would be the objects of these representations.

[43]In Bolzano's opinion we can have, for instance, intuitions of our subjective ideas—if we do, they are called clear. See *Wissenschaftslehre*, §280.

[44]Bernard Bolzano, *Von der mathematischen Lehrart*, §6.

[45]Cf. *Von der mathematischen Lehrart*, §6.4: "But as certain as it is that ideas of the form 'This, which now occurs in me' are singular ideas, just so is it certain that among these there are at least some which are completely simple. For if we suppress the thought of any additions like 'which

which are real, namely, the changes which occur in us at a particular moment, and—given the causal nexus of real things—we are also justified in inferring from the presence of an intuition in our mind the existence of some real thing which is its mediate cause.

There are doubtless some troubling aspects of Bolzano's account of intuitions. Noteworthy, however, is the fact that his characterization of them as simple representations with exactly one object makes no reference to the mind or its faculties; it is, rather, objective, applying in the first instance to representations in themselves. This methodological scrupulousness allows him to give an objective characterization of concepts as representations which are not intuitions and which contain no intuition as a part; and further, to define purely conceptual propositions as those which contain only concepts; and finally, purely conceptual sciences as those in which all the propositions are purely conceptual. In this group, notably, belong many of the central branches of mathematics, including the theories of collections or sets, numbers, quantities, functions, time and space. As purely conceptual sciences, these parts of mathematics have only conceptual resources at their disposal: namely, simple (or primitive) concepts, which are distinguished only by their types and their extensions; other concepts defined in terms of these; axioms; and theorems deducible from these. In short: mathematics is understood as a formal or rather formalized system in almost all respects like those described some 80 years later by Hilbert and others. The grounds of purely conceptual propositions, furthermore, can lie only in other purely conceptual propositions. Thus the underpinnings of Bolzano's definitive exclusion of intuitions from these branches of mathematics.

The distinction between purely conceptual propositions and sciences and those which contain intuitions is for Bolzano an extremely important one, one which effectively takes the place of the traditional division between the *a priori* and the

occurs in me just now, 'which I just now see, hear, or feel', or 'which I am now pointing at with my finger', etc., the bare idea designated by the word 'this', is certainly a completely simple idea. But the object which it represents remains throughout the same single one, whether we think the additions or not. For, if we consider them more closely, all these additions express no more than certain attributes which that single object which we just now represent possesses precisely because it is this one and no other; in such a way that our idea does not become restricted to that single object only by means of these additions, but rather becomes redundant. Even were it the case that with many, indeed with all, of these ideas, similar additions were made involuntarily in the mind, so that, instead of simple ideas of the form 'This', only complex ideas like 'This, which I just now perceive in myself', or 'This, which I just now see', or 'This red', 'This sweetness', 'This sourness', 'This feeling of pain' were to be met with in our consciousness, it has still been shown that there are simple, singular ideas, whether we be conscious of them in isolation or only when they already occur in combinations. But as no composite idea comes to be without its being generated from its simple parts by means of a characteristic activity of our soul, there is no doubt that each one of the just mentioned simple components exists in isolation for a longer or shorter time in our soul before it enters into combinations."

empirical. These two distinctions, Bolzano admits, very nearly coincide, since, he thinks, purely conceptual propositions can often be known without the help of what is called experience.[46] But the distinction which interests Bolzano is, at bottom, not an epistemological one; hence the pains he took to characterize it without reference to a knowing subject. To say that a proposition or a science is purely conceptual is a statement concerning its content, not about the means in which we might come to know it. It is perfectly conceivable, for instance, that we could come to know purely conceptual propositions with the help of experience—Bolzano cites as examples various propositions of number theory and the inverse square law of gravitation.[47] It is not a question of knowing, but rather only when we wish to set out the objective ground of a purely conceptual proposition that we are obliged to use purely conceptual means. And—perhaps more importantly—it is possible to hold a purely conceptual proposition to be true erroneously, and also to hold it to be true on merely probable grounds. Indeed, Bolzano thought that in many cases this was how things stood, since in geometry the objective grounds of most propositions had never been distinctly set out and in analysis many false propositions were widely accepted. To Kant's claim that the *a priori* went hand in hand with certainty, infallibility, and felt necessity,[48] Bolzano replied with the authority of the practitioner:

> If we have not tested the truth of a proposition either by experiment, or by repeated checking of its derivation, we do not give it unqualified assent, if we are at all sensible, no matter what the Critical Philosophy may say about the infallibility of pure intuition... Does not experience teach us that we make mistakes in mathematical judgments, and that we make these mistakes all the more easily the more we trust what that philosophy calls by the high-sounding name of pure intuition?[49]

A proposition can be purely conceptual and judged true by us; but this by no means entails that it is unrevisable or beyond question. This leads me to a final point of comparison. Kant, with his emphasis on construction and his distrust of speculation, has often been hailed as a forerunner of more exact methods in mathematics and logic, while Bolzano, with his realm of ideas and propositions

[46] Bernard Bolzano, *Wissenschaftslehre*, §133.

[47] Bernard Bolzano, *Von der mathematischen Lehrart*, §7.

[48] See, for example, *Critique of Pure Reason*, A424-5/B452: "In mathematics ... [the employment of the sceptical method]... would, indeed, be absurd; for in mathematics no false assertions can be concealed and rendered invisible, inasmuch as the proofs must always proceed under the guidance of pure intuition and by means of a synthesis which is always evident."

[49] Bernard Bolzano, *Wissenschaftslehre*, §315,4; cf. *Beyträge*, Anhang, §10: "... the vaunted certainty of mathematics gradually disappears as experience fails us... "

in themselves, has acquired the reputation of a logical Plato and a purveyor of metaphysical extravagances. But if we compare what the two have to say about our knowledge of the mathematical sciences, surely it was Bolzano who had his feet on the ground.

Bolzano and Lagrange

When many would have been ready to think about retirement, Lagrange accepted an offer to teach analysis at the newly founded *École Polytechnique*. His activity there, especially his production of texts based on his lecture notes (the *Théorie des fonctions analytiques* and the *Leçons sur le calcul des fonctions*), was to play an important role in the early development of modern functional analysis. Just as early twentieth-century mathematicians spoke of Cantor's paradise, mathematicians of the nineteenth century might well have spoken of Lagrange's: an elegant theory of functions which lacked for nothing except an acceptable foundation. Cauchy's intensive study of Lagrange's analysis is well known, as are the criticisms and responses, set out in his *Cours d'analyse* (again derived from lectures at the *École Polytechnique*) which officially opened the modern period of analysis. Lagrange's impact on Bolzano, another key figure in the history of analysis, is not perhaps so well known, but was equally significant. Bolzano, like Cauchy, carefully studied Lagrange's texts from the moment of their appearance, learning from them many points of technique and general conception while noting at the same time grave problems of logical sequence and organization. Like Cauchy, too, Bolzano developed an alternative *Theory of functions*, one which, however, was not to be published during his lifetime.

Lagrange's texts are, when we consider the circumstances of their creation, something of a marvel. Called upon to teach analysis to engineers in training, he revived ideas he had had some years earlier concerning the foundations of the subject and worked them through at great speed. Given the haste of their composition, and their intended audience, these texts are remarkably good. The usual topics—derivatives, differential equations, applications, etc.—are discussed, and the presentation is generally quite comprehensible even for beginners. Pedagogically, by most accounts, Lagrange's approach was a success, in marked contrast to that of Cauchy (whose presentation of analysis was both too time-consuming and overly refined for an engineering school[50]). More interesting for mathematicians were Lagrange's attempts to introduce more rigour into his presentation, in the course of which, among other things, he worked out some good formalizations of key concepts and proof techniques. Not a timid mathematician or teacher, he

[50]Cf. Bruno Belhoste, *Cauchy : un mathématicien légitimiste au XIXe siècle* (Paris: Belin, 1985), 78 f.

was not afraid to make a few mistakes; indeed, his work abounds in them. Often they are interesting mistakes, the sort which require both hard work and insight to clear up.

In his *Contributions to a better founded presentation of mathematics* of 1810, Bolzano distinguishes between two kinds of presentation of a science, the strictly scientific and the practical.[51] The latter—which includes many features dictated by pedagogical requirements—is by far the more important in his view: for sciences are presented in books above all in order to be communicated so that the knowledge contained in them may be used to the human lot. He nevertheless devoted himself to the first type of development. In part, this came from the conviction that in order to write a truly excellent practical treatise one had first to have a rigorously scientific grasp of a science.[52] But there were other reasons as well: instilling logical order into a science allows us to understand its concepts more easily and accurately, and also to discover new truths, for "more can be inferred from the first ideas when they are correctly and distinctly set out than when they are scattered about in confusion." Finally, the scientific presentation is precisely the sort which does the most to foster the development of a thorough way of thinking, thus, the best for philosophical training.[53]

Bolzano's *Function theory* is not a practical treatise like those of Lagrange, and hence not a direct response to them except insofar as they were presented as or taken for a strict foundation (as indeed they were by many of Bolzano's contemporaries[54]). Clearly Bolzano greatly valued Lagrange's work, and learned much from it. In responding to it, he sought above all to conserve as much of its structure as was compatible with his methodology. It was mostly a matter of suitably determining the meanings of the concepts involved, making the statements of the theorems more accurate, and (in his apt phrase) "bringing truths which are already known to us into such an order as they themselves prescribe."[55] Lagrange had a real talent for stating significant theorems, Bolzano for grounding them. Thus despite Bolzano's trenchant criticisms of Lagrange,[56] I think it possible to view

[51] Bernard Bolzano, *Beyträge*, I, §19.

[52] *Ibid.*

[53] Bernard Bolzano, *Betrachtungen über einige Gegenstände der Elementargeometrie* (Prag, 1804), Vorrede.

[54] See Bernard Bolzano, *Miscellanea Mathematica* 13 ed. Bob van Rootselaar and Anna van der Lugt Bernard Bolzano-Gesamtausgabe (Stuttgart-Bad Cannstatt; Fromann-Holzboog, 1998) s.2 t.8 for a sample of Bolzano's reading of contemporary texts on analysis, many of which were heavily indebted to Lagrange.

[55] Bernard Bolzano *Wissenschaftslehre*, §2.

[56] See, for example, *Miscellanea Mathematica* 13 ed. Bob van Rootselaar and Anna van der Lugt Bernard Bolzano-Gesamtausgabe (Stuttgart-Bad Cannstatt; Fromann-Holzboog, 1998) s.2 t.8/1, 76 f. for Bolzano's comments on Lagrange's *Leçons*. Occasionally, Bolzano's patience wears thin.

the relations between the two as being primarily a collaboration, undertaken by Bolzano at least partly with the aim of improving later generations of textbooks.

The comparison of function theory in Lagrange and Bolzano offers a rich mine of material, one which repays careful study. Here, I will be able to discuss only a small sample. First, I will attempt to sketch as briefly as possible the general contours of Lagrange's analysis and Bolzano's response. Then I will turn to the concept of continuity for a more detailed discussion.

Lagrange's analysis is a theory of *functions*: the curves and differentials prominent in earlier treatments of the subject are gone. Analysis appears as an autonomous discipline prior to geometry, which is placed along with mechanics among the applications of this more general science. Functions are, primarily, finite analytic expressions, indicating the dependence of a quantity on others according to a given law.[57] The central assumption is that every function f(x), its variable x changed by an increment i, can be developed in a power series of the form:

$$f(x+i) = f(x) + ip + i^2q + i^3r + \cdots \qquad (1)$$

This is not a numerical equation. As Lagrange noted in the *Leçons sur le calcul des fonctions*, "the development of $f(x+i)$ is only generally true *insofar as one does not give x specific values*."[58] It is, instead, a relation of what he calls algebraic form. Lagrange offers a rather flimsy proof of the assertion that every function can be developed in such a series; but a proof was not really required, since he and most other analysts of the time knew how to derive the development for almost all of the functions they had occasion to use. The derived function f' of f is then defined in terms of the development: it is the coefficient $p = p(x)$. The derived function of $p(x)$, or the second derived function of f, f'' is defined similarly in terms of the power series for f', and, after some free-handed manipulations of series, Lagrange is able to show that the development has the form of the Taylor series for f at x:

$$f(x+i) = fx + if'(x) + i^2\frac{f''(x)}{2!} + \cdots + i^n\frac{f^{(n)}(x)}{n!} + \cdots \qquad (2)$$

This representation, along with various algebraic manipulations, then allows him to derive the standard results of the calculus—derivatives of polynomials and rational functions, definitions and derivatives of the exponential, logarithmic, and trigonometric functions, derivation of the binomial series, etc. —expressed in terms of his derived functions. The problems of pure analysis are dealt with on this level, without reference to specific numerical values. Only afterwards, when it is a question of

[57] Joseph-Louis Lagrange, *Théorie des fonctions analytiques*, *Œuvres de Lagrange*, t. 10 (Paris, 1884), 10.

[58] Joseph-Louis Lagrange, *Leçons sur le calcul des fonctions*, *Œuvres de Lagrange*, t. 9 (Paris, 1881), 43, emphasis added.

applying analysis—for instance, to geometrical problems like finding tangents—, does Lagrange consider the worth of his series as numerical equations. As usual, he arrives at some very significant theorems, notably the Taylor series with the so-called Lagrange form of the remainder:[59]

$$f(x+i) = fx + if'(x) + i^2\frac{f''(x)}{2!} + \cdots + i^n\frac{f^{(n)}(x)}{n!} + i^{n+1}\frac{f^{(n+1)}(x+\mu i)}{(n+1)!} \quad (3)$$

for some μ with $0 \leq \mu \leq 1$.

But since he did not build numerical representation into his analysis, it comes as little surprise that his proofs are rarely adequate and often incoherent.[60] They nevertheless contained many things of value for the careful reader.

Bolzano's *Function Theory*, his detailed response to Lagrange, is in one sense well known, another not. Well known, because in many respects it closely resembles the standard approach to real analysis associated with the names of Cauchy and Weierstrass. Not so well known in that it was only published in 1930 and, not having had any detectable influence on nineteenth-century mathematics, has received relatively little attention from historians of mathematics (and next to none from philosophers).

Bolzano begins, as is appropriate for philosophical inquiry, with questions of meaning. What is a real number? a function? a continuous function? what does it mean to say that an infinite series represents a given function? and so on. His definition of function, for example, is considerably wider than Lagrange's, corresponding to the standard definition of classical analysis. Real-valued functions for him are simply mappings; algebraic form in Lagrange's sense—if it exists at all—is a derivative property of these. Series, for their part, are considered not only as developments of analytic functions, but as independent mathematical objects in their own right.[61] The sum of an infinite series is defined in terms of convergence, and a series is said to represent a function on a domain iff it converges to the value of the function for each real number in the domain. These decisions are dictated in advance by the requirement that analysis be applicable: for if (as with Lagrange) no reference were made to precise numerical representation in the foundations, it is difficult to see how it might be established afterwards. Lagrange's general claim concerning the development of a function in a power series—already false in its own terms, as Bolzano pointed out[62]—has to be abandoned all the more for the

[59] *Leçons*, 100.

[60] I have discussed these matters at greater length in the paper: "Remaking mathematics: Bolzano reads Lagrange," *Acta analytica* 18(1997)51-72.

[61] Craig Fraser, "The calculus as algebraic analysis: some observations on mathematical analysis in the 18th century," *Archive for History of Exact Sciences* 39 (1989) 317-335, has underlined this shift in understanding of series in the transition from 18th to 19th century analysis more generally.

[62] Bernard Bolzano, *Functionenlehre*, ed. K. Rychlik (Prag, 1930), II, §91.

more general class of functions under consideration. Hence, in order to ground Taylor's theorem, additional conditions have to be specified, further definitions set out: for instance, the continuity of a function, the derivative of a function, etc. Only after all this work, and the proof of the necessary lemmas, is Bolzano in a position to give a precise statement and proof of the theorem with which Lagrange began his work on function theory.

Continuity

To illustrate the nature of Bolzano's transformation of Lagrange's analysis in more detail, I will now consider his exposition of the concept of continuity. Lagrange's notion of function was, as noted, still very close to that of Euler's *Introductio*—functions were associated and often identified with finite analytical expressions. These functions—save at isolated exceptional values—are continuous, and it is perhaps not surprising that this banal property received so little attention from eighteenth-century analysts. With Bolzano, the widening of the function concept meant that this property had to be precisely specified and investigated.

As various historians have pointed out, the lack of a theory of continuity in eighteenth-century analysis was not due to a technical shortcoming on the part of Lagrange or his contemporaries.[63] Although Lagrange had no theory of continuous functions, important elements of such a theory were present in his work, waiting to be found. To begin with, Bolzano could—and quite likely did—find a perfectly adequate definition of continuity at a point in Lagrange's treatises. But being able to produce such a formulation is one thing; arriving at a full appreciation of its logical role in the foundations of functional analysis quite another.

In No. 6 of the *Théorie des fonctions analytiques*, Lagrange attempted to find a general error estimate for his power series expansions as follows: We have

$$f(x+i) = f(x) + pi + qi^2 + ri^3 + \cdots \tag{4}$$

Let us write

$$iP = pi + qi^2 + ri^3 + \cdots$$
$$iQ = qi^2 + ri^3 + \cdots$$

for the remainders. Lagrange now asks us to consider the curve with i as abscissa and iP as ordinate and claims that

> ...since the remainders iP, iQ, iR [...] are functions of i which become zero, by the very nature of the development, when $i = 0$, it

[63] See, e. g. A. P. Youschkevitch, "The concept of function up to the middle of the 19th century," *Archive for History of Exact Sciences* 16(1976)37-85, 71-2.

> follows that [. . .]this curve will cut the axis at the origin of the abscissae [. . .] and the course of the curve will necessarily be continuous from this point on; therefore it will approach the axis little by little before cutting it, and will approach it, consequently, by a quantity less than any given quantity, so that one can always find an abscissa i corresponding to an ordinate less than a given quantity, while to each smaller value of i will also correspond ordinates less than the given quantity.[64]

One can in particular, he claims, take i so that iP will be less than fx, which was to be shown.

In his description of the behaviour of iP near zero Lagrange has a precise statement of

$$\lim_{i \to 0} iP = 0$$

one which is readily adaptable for expressing the continuity of a function at a point. The problem, as Bolzano pointed out in the *Contributions*, is that Lagrange seeks to derive the continuity of the function from the continuity of its graph, a proof technique which simply moves us in a small circle.[65] But the worthlessness of the proof does not detract from the usefulness of the analytical formulation, one which Bolzano himself adopted: for him, a function is continuous at a point iff it satisfies Lagrange's property there; and on an interval iff it is continuous at every point in the interval.[66]

Another important distinction marked by Bolzano, that between what today are called pointwise and uniform continuity on an interval, may also have been prompted by his careful reading of Lagrange.[67] Lagrange began the ninth of his *Leçons sur le calcul des fonctions* with a proof that a function whose derivative is positive on an interval $[a, b]$ with $f(a) = 0$ will be positive on the interval.[68] The proof begins with the observation that, based on the series development of f, we have:

$$f(x + i) = f(x) + i \left[f'(x) + V \right] \tag{5}$$

[64] Lagrange, *Théorie des fonctions analytiques*, No 6.

[65] Cf. "Remaking mathematics: Bolzano reads Lagrange," 68-70.

[66] *Functionenlehre*, I, §2; in a note, Bolzano credits Lagrange for using the word "continuity" in this sense.

[67] Bolzano's grasp of the concept of uniform continuity is discussed in detail in Paul Rusnock and Angus Kerr-Lawson, "Bolzano and uniform continuity," *Historia Mathematica* (forthcoming).

[68] We know that Bolzano studied these pages of Lagrange quite carefully—see, for example, *Rein analytischer Beweis*, Vorrede, V, for some detailed remarks. Also *Miscellanea Mathematica* 13, 86 f).

where V is a function which becomes zero for $i = 0$. (Lagrange scrupulously notes that V is a function of x *and* i.) Because V goes to zero with decreasing i, he continues,

> one can always assign i a value such that the corresponding value of V will be less than a given quantity in absolute value, and such that the value of V will also be smaller [than the given quantity] for smaller values of i.[69]

Hence given D, we can choose i sufficiently small to guarantee that

$$f(x+i) - f(x) = i\left[f'(x) + V\right] \tag{6}$$

is bounded by the values

$$i\left[f'(x) \pm D\right] \tag{7}$$

Now, he says, this holds for *any* value of x, which permits us to write the array of inequalities

$$i[f'(x) - D] < f(x+i) - f(x) < i[f'(x) + D]$$

$$i[f'(x+i) - D] < f(x+2i) - f(x+i) < i[f'(x+i) + D]$$

$$i[f'(x+2i) - D] < f(x+3i) - f(x+2i) < i[f'(x+2i) + D]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$i[f'(x+(n-1)i) - D] < f(x+ni) - f(x+(n-i)i) < i[f'(x+(n-1)i) + D]$$

Adding, we get $f(x+ni) - f(x)$ confined between the limits:

$$if'(x) + if'(x+i) + if'(x+2i) + \cdots + if'(x+(n-1)i) \pm niD \tag{8}$$

a result which allows him, after a few further steps, to complete his proof.

By first noting and then later ignoring the fact that V is a function of x as well as i, Lagrange clearly points to an assumption which—if not for him, at least for Bolzano—requires proof. If we divide Lagrange's inequalities by i, we see that this assumption amounts to saying that f' is the *uniform* limit of the differential quotient on the interval. When, in II, §27 of his *Function Theory*, Bolzano set out to prove a version of the fundamental theorem of the calculus, he did his best to fill this gap, diligently announcing and attempting (unsuccessfully) to prove that if a

[69] *Leçons*, p. 87.

function is continuously differentiable on a closed interval, then f' is the uniform limit of the differential quotient there.[70]

In a manuscript containing emendations and additions to the *Function Theory*, Bolzano announced the analogous result for functions continuous on a closed interval (usually called Heine's theorem[71]): if a function is continuous on a closed interval, then it is uniformly continuous there. In this case, although he did not provide a satisfactory proof, Bolzano did manage to provide a useful fragment of one.[72] In I, §13 of the work itself, he had noted that the same does not hold on an open interval:

> Merely because a function $F(x)$ is continuous for all values of its variable x lying between a and b, it does not follow that for all x between these values there is a fixed number e which is small enough so that one can claim that Δx never has to be taken smaller in absolute value than e in order to ensure that the difference $F(x + \Delta x) - F(x)$ will turn out to be smaller than $1/N$.[73]

He proves this claim by considering the behaviour of $f(x) = 1/(1 - x)$ on an interval $(a, 1)$.

Here, once again, to a gap indicated by Lagrange there corresponds a fragment of theory in Bolzano's analysis.

Armed with his precise definition of continuity, Bolzano undertook a careful examination of the properties of continuous functions: investigating the relations between continuity and monotonicity, differentiability, boundedness, the occurrence and distribution of maxima and minima and so on. In addition to the intermediate value theorem, he proves that a function continuous on a closed interval is bounded and assumes its extrema there, that a continuous function may assume either local or global extrema an arbitrary number of times on any interval, that a continuous function need not be monotone on any subinterval of a given interval, and, famously, that a function may be continuous on an interval and yet fail

[70]Bolzano, *Functionenlehre*, II, §27: "Theorem: If a function Fx has a derivative $F'x$ in both directions for all x between a and $a + h$, at $x = a$ at least a derivative in the same direction as h and at $x = a + h$ in the opposite direction; if, moreover, this derivative obeys the law of continuity for all the just named values of x; then there must be a number e which is small enough so that we can claim that the increment Δx never has to be taken smaller than e in order for the difference $(F(x + \Delta x) - Fx)/\Delta x$ to turn out to be less than a given fraction in absolute value, provided only that x and $x + \Delta x$ do not lie outside a and $a + h$."

[71]Though Heine's proof is actually due to Dirichlet: cf. Pierre Dugac, "Sur la correspondence de Borel et sur le théorème de Dirichlet-Heine-Weierstrass-Borel-Shoenflies-Lebesgue," *Archives internationales d'histoire des sciences* 39 (1989) 69-110.

[72]Cf. Rusnock and Kerr-Lawson, "Bolzano and uniform continuity."

[73]Bolzano, *Functionenlehre* , I, §13.

to have a derivative on a set of points dense in the interval. The proofs, involving the construction of ingenious counterexamples and the deft use of topological and quantificational concepts, announce the beginnings of a new kind of mathematics undreamt of by Lagrange. We have here a cluster of theorems which is surely among Bolzano's finest work as a mathematician, one which clearly exemplifies the merits of his methodology. There is, I think, little if anything to compare with it in the earlier literature.

Consider, for example, his proof that a function, f, continuous on a closed interval $[a, b]$ attains its maximum on the interval. His proof involves the following three results: [i] a function which is continuous on a closed interval is bounded there;[74] [ii] A bounded set of real (he calls them "measurable") numbers has a least upper bound;[75] and [iii] if a function f is continuous on a closed interval $[a, b]$ and comes arbitrarily close to a value C on the interval, then $f(x) = C$ for some x in $[a, b]$. The third of these theorems is proved along the following lines.[76] Suppose f comes arbitrarily close to the value C on $[a, b]$. Then either $f(x) = C$ for some $x \in [a, b]$, or there exist $x_1, x_2, \ldots$ such that $|C - f(x_n)| < 1/n$, for $n = 1, 2, 3, \ldots$ By the Bolzano-Weierstrass theorem,[77] the x_i will have a limit point, c, in $[a, b]$. Since f is also continuous at c, it follows as an easy consequence that $f(c) = C$. The main result is then proved by noting that (by [i]) the values of f are bounded on $[a, b]$, and hence (by [ii]) the set of values has a least upper bound, M. Since M is the least upper bound for f, f must either attain or come arbitrarily close to M on $[a, b]$. By the theorem [iii] cited above, it follows that $f(x) = M$ for some x in $[a, b]$.

This, as Bolzano himself might have remarked, is a proof of the obvious. Lagrange or any other analyst would certainly have allowed himself to use this result without proof, regardless of whether or not it was thought to require one. But by attempting to objectively ground propositions which everyone already knew to be true, Bolzano put himself in the position Thales had found himself in: he could now discover many results which formerly were unattainable because the fine-grained conceptual structure had not been set out. When we see the clear outline of a topological foundation for analysis in these few pages of Bolzano, we understand just how powerful a position this was.

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[74] Bolzano, *Functionenlehre*, I, §20 f.

[75] This was proved in the 1817 *Rein analytischer Beweis* (§12). For a proof in Bolzano's theory of measurable numbers, see *Reine Zahlenlehre*, §109 (Bolzano, Gesamtausgabe (Stuttgart-Bad Canstatt: Fromann-Holzboog, 1976), Reihe II, Bd. 8, 156 f.)

[76] Bernard Bolzano, *Functionenlehre*, I, §22.

[77] Bolzano refers to a proof of this result in the theory of measurable numbers (*Functionenlehre*, I, §20), but it has not been found there. There is no good reason to suspect that he did not have one, however.

Conclusion

As I remarked at the beginning of this essay, Bolzano saw the philosophy of mathematics as simply a matter of foundations. He would not as a consequence have considered his critical response to Kant to belong to the philosophy of mathematics properly speaking. Indeed, according to his view, it would have been impossible to discuss Kant's philosophy of mathematics because Kant had never done any work in the area—he had never, that is, attempted to provide a rigorous proof for any mathematical proposition. Bolzano's demarcation of the philosophy of mathematics will no doubt appear odd to many philosophers today—surely, it will be objected, Kant had something to say about the philosophy of mathematics? No doubt this is true, and Bolzano's usage too narrow. I nevertheless think that Bolzano's point of view has something to be said for it, as I hope my discussion of his response to Lagrange has indicated. In seeking to ground the theorems of analysis, Bolzano was compelled to investigate a number of topics which would later prove most interesting to philosophers of mathematics, among them set theory, certain aspects of general topology, and the logic of quantificational concepts. Beyond this, his work in the philosophy of mathematics led to significant changes in mathematics itself, changes whose value cannot seriously be contested. In Bolzano's understanding, philosophy of mathematics is an active, creative force, one aimed not at understanding a static object from a distance but at changing a dynamic one from within. To be sure, if we follow him in his usage, it will turn out that most philosophy of mathematics has been done by mathematicians rather than philosophers. What is more, they will turn out to have been philosophizing for the most part without being aware of it. In this respect, and doubtless in others too, Bolzano's view is inconvenient. Like a number of other inconvenient opinions, though, it might contain some important truth.