Bolzano and the Traditions of Analysis

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§**1**

Russell's discussion of analytic philosophy in his popular *History* begins on a surprising note: the first analytic philosopher he mentions is . . . Weierstrass. His further remarks—in which he discusses Cantor and Frege, singling out their work in the foundations of mathematics—indicate that he thought that the origin of modern philosophical analysis lay in the elaboration of modern mathematical analysis in the nineteenth century [13, 829-30]. Given the markedly different meanings attached to the word "analysis" in these two contexts, this juxtaposition might be dismissed as merely an odd coincidence. As it turns out, however, modern philosophical and mathematical analysis are rather closely linked. They have, for one thing, a common root, albeit one long since buried and forgotten. More important still, and apparently unknown to Russell, is the circumstance that one individual was instrumental in the creation of both: Bolzano.

Russell's account could easily leave one with the impression that analytic philosophy had no deep roots in philosophical tradition; that, instead, it emerged when methods and principles used more or less tacitly in mathematics were, after long use, finally articulated and brought to the attention of the philosophical public. A most misleading impression this would be. For right at the beginning of the reconstruction of the calculus which Russell attributed to Weierstrass we find Bolzano setting out with great clarity the methodology guiding these developments in mathematics—a methodology which, far from being rootless, was developed in close conjunction with Bolzano's usual critical survey of the relevant philosophical literature. But not merely that: for Bolzano also put this methodology to work with considerable skill and precision, developing many of the central elements of the new mathematical analysis while Weierstrass was still in short pants. Far from an unwitting carrier of modern philosophical analysis, Bolzano's mathematical research was just one of several quite conscious *applications* of it in his thought.

Bolzano's influence on Weierstrass and his circle—and thus, at second remove, on Russell—has been firmly established.[1] There is good reason, then, to include Bolzano in the history of analytic philosophy and not just as an isolated, uninfluential precursor (§6). My main concern here, however, is not the transmission of Bolzanian philosophy, but rather its roots, in particular the question of why Bolzano thought that both philosophical and mathematical analysis were ripe for reform. My discussion begins in earnest with Descartes (§3), who stands close to the beginning of the modern evolution of the two analyses. Before this, I present a sketch of some earlier developments (§2).

§**2**

The ancient method of geometrical analysis, we are informed by Pappus (who also furnishes us with a "Treasury" of examples of the technique), is that of the "solution backwards":

> [I]n analysis we assume that which is sought as if it were already done, and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by retracing our steps we come upon something already known or belonging to the class of first principles.[2]

Synthesis, on the other hand, is the solution forwards, proceeding from the first principles or their already proven consequences to obtain new theorems.

Philosophy, for its part, also laid claim to a method of analysis. Here is how Aristotle described things at the beginning of the *Physics*:

> When the objects of an inquiry, in any department, have principles, conditions, or elements, it is through acquaintance with these that knowledge, that is to say scientific knowledge, is attained. For we do not think that we know a thing until we are acquainted with its primary conditions or first principles, and have carried our analysis as far as its simplest elements. Plainly therefore in the science of nature,

[1] See, for instance, [21, 95-6], [15, 77], [29, 61].

[2] Quoted after [11, II, 400].

> as in other branches of study, our first task will be to try to determine what relates to its principles.
>
> The natural way of doing this is to start from the things which are more knowable and obvious to us and proceed towards those which are clearer and more knowable by nature [1, I, 1 (184^a1 f.)].

Here we find two senses of analysis which have shown real endurance: one of resolving complexes into their simple parts; and a second of progressing from that which we know towards principles—that is, the (generally unknown) grounds or causes of the known. Aristotle also spoke of this second sense as being concerned with the question of *why* something is so rather than *whether* it is.[3]

There are two logical constraints governing the choice of first principles: 1) they must be indemonstrable; and 2) it should be possible to derive all the truths of the science from them. These requirements were given some substance by setting out a theory of demonstration. But as the fine structure of universals was not considered, the problem was left logically underdetermined. Hence the non-logical constraints which Aristotle felt compelled to pile on: first principles must be ontologically primary, expressing the essences of things; they must be prior to and better known than the derived truths; they must be the grounds or causes of the derived truths; they must be farthest from what is given to the senses, and so on.[4]

Geometrical analysis, like its philosophical counterpart, was concerned with a movement towards first principles, but in a different sense. For in geometry—at least in its mature period—, the first principles were (generally speaking) known in advance: they, along with their already proven consequences, are what is called "the given". The goal of geometrical analysis was not, as in philosophical analysis, to *discover* the first principles, but rather to attempt to integrate a proposition into an existing deductive framework. Returning to Aristotle's distinction, we could say that geometrical analysis is concerned primarily with the question of *whether* a given proposition is true. The why will, indeed, be answered at the same time, but only secondarily. For analysis simply shows us how to graft a (true) proposition onto a pre-existing tree; and the tree already contains most of the why. Geometrical analysis is a local searching within a framework whose global structure is for the most part fixed in advance.

I say for the most part because there must have been occasions when geometrical analysis led mathematicians to add to the number of first principles. For

[3]E.g. [5, 78^a22 f.].

[4]E.g. [5, I, 2 (71^b19 f)]

example, when it was found that the technique of *neusis* constructions could solve some important problems which had so far resisted solution (e.g. the trisection of the angle), the assumption that this could be done was incorporated as one of the "givens".[5] Overall, however, geometrical analysis was characterized by a considerable stability in the given. The addition of new first principles would be rare. Those already recognized, which would serve for the solution of a good many problems, were public knowledge, codified in textbooks. And the same could also be said for the rules of the game: that is, the inferences one was allowed to draw in the course of an analysis.

With philosophy, things were quite otherwise. The first principles, far from being public knowledge, were the target of analytic inquiry. And the rules guiding inquiry were anything but fixed, something which Aristotle frankly admitted at the beginning of his treatise on the soul [2, I, 1 (402^a10 f.)].

This difficulty explains the marked difference in presentation between Aristotle and Pappus. Aristotle expends a good deal of energy attempting to determine what a good first principle would look like. His characteristic decision to tie logical structure to ontology solves the problem in a way: for things are, and if they have essences, then those essences are given, at least somehow. Considered at a safe distance, however, we can see that this check is still in the mail.

Pappus, by contrast, had his first principles in hand. He can simply give a loose, rather free description of what analysis is, and then move on to what really interests him: the technical details. As he explains, he is not so much interested in presenting an abstract discussion of methodology as providing a manual for advanced research:

> The so-called "treasury of analysis" is, to put it shortly, a special body of doctrine provided for the use of those who, after finishing the ordinary Elements, are desirous of acquiring the power of solving problems which may be set them involving [the construction of] lines, and it is useful for this alone.[6]

Philosophical discussion has, in large part, focused on the description of analysis, neglecting what Mahoney has aptly called the "mathematical toolbox" which

[5] On *neusis* constructions, see [11], I, 235 f. Although new methods were admitted, Greek geometers were generally very scrupulous about classifying problems according to the types of constructions required for their solution: hence the famous division into *plane*, *solid*, and *linear* problems, depending on whether the solution required only straightedge and compass, or in addition to these conic sections; or still further curves on top of these.

[6] Quoted after [11, II, 400].

forms the bulk of Book 7 of Pappus' *Collection* [19, 320]. But this latter sense of "analysis", which would later become primary, seems already to be established in antiquity—one, that is, referring not so much to a method as to a body of techniques for problem solving.

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Despite significant differences between mathematical and philosophical analysis in antiquity, the similarities were clearly sufficient to justify using the same name for both. Things became more complicated after the intervention of the renaissance mathematician François Viète. Viète's "Analytic Art" set out many of the essentials of equation theory and its application to a variety of problems. Viète claimed no particular novelty for his work, presenting it instead as largely exegetical in character, restoring to its previous greatness the ancient method of analysis which had become corrupted by passing through the hands of the unwashed.

> Behold, the art which I present is new, but in truth so old, so spoiled and defiled by the barbarians, that I considered it necessary, in order to introduce an entirely new form into it, to think out and publish a new vocabulary, having gotten rid of all its pseudo-technical terms . . . lest it should retain its filth and continue to stink in the old way, but since till now ears have been little accustomed to it, it will be hardly avoidable that many will be offended and frightened away at the very threshold. And yet underneath the Algebra or Almucabala which they lauded and called "the great art," all mathematicians recognized that incomparable gold lay hidden, though they used to find very little.[7]

In Viète's work, the link with ancient geometrical analysis is quite clear: in a very precise sense, one assumes the solution to be made, assigning letters to the knowns and the unknowns, and uses the algebraic techniques to reduce the problem to "the given" (i.e. specific numbers, or constants/parameters) when this is possible. Thus, for example, a geometrical problem might lead to a quadratic equation like $ax^2 + bx = c$, where x, by the very writing down of the equation, is in a certain sense "assumed as if it were known", and where a, b, c are "given" parameters determined by the geometrical situation.[8] The reduction of this equation

[7] Quoted after [26, 318-19].

[8] Here, for the sake of familiarity, I have used Descartes' notation rather than Viète's.

by publicly sanctioned inferences corresponds naturally to what occurs in geometrical analysis. Even the investigation of the conditions required for the possibility of a solution (the *diorismos*, in ancient terminology) has its place here: if a real solution is sought, for instance, a brief calculation shows that one will require $b^2 + 4ac \geq 0$.

But the link was soon forgotten. Instead, it was the growing body of technique which caught the eye. Christian Wolff's entry on analysis in his *Mathematisches Lexicon* of 1716 reflects this change in emphasis:

> *Analysis*, the art of resolution, is a science for solving questions as yet unanswered, or for discovering unknown truths from certain known ones. As today for the most part specious arithmetic, algebra and the differential calculus of Herrn von Leibniz are used for this, it is customary for these methods to be called collectively analysis.[9] [8, Art "Analysis"]

Much the same can be found in d'Alembert's *Encyclopédie* article: "Analysis," he writes, "in order to solve problems, employs the assistance of algebra, or general calculus of quantities: hence these two words, *analysis*, *algebra*, are frequently regarded as synonyms."[10] [3]

A later dictionary entry by d'Alembert on a related topic shows how far things had moved by the end of the eighteenth century. There, he describes the analytic and the synthetic method in geometry as follows: the analytic, which is also the modern, method, uses algebra to solve problems and prove theorems; the synthetic, or old-fashioned method, uses diagrams and constructions[11] [6, 48]. These senses of analysis and synthesis are still quite prominent, perhaps even primary,

[9] Analysis, die Auflösungskunst ist eine Wissenschaft die verborgenen Fragen aufzulösen, oder aus einigen erkandten Wahrheiten andere noch unbekandte zu erfinden. Da man nun heute zu Tage meistens die Buchstaben-Rechen-Kunst, die Algebra, und die Differential-Rechnung des Herrn von Leibnitz dazu brauchet; pfleget man diese Erfindungs-Kunst zusammen die Analysis zu nennen.

[10] "L'analyse, pour résoudre les problêmes, emploie le secours de l'algèbre, ou calcul des grandeurs en générale: aussi ces deux mots, *analyse*, *algèbre*, sont souvent regardés comme synonymes."

[11] "*Méthode analytique*, en *Géometrie*, est la méthode de résoudre des problêmes, & de démontrer les théorêmes de Géometrie, en y employant l'analyse ou l'algèbre. [...] Cette méthode est opposée à la méthode appellée *synthétique*, qui démontre les théorêmes, & résout les problêmes, en se servant des lignes mêmes qui composent les figures, sans représenter ces lignes par des noms algébriques. La méthode synthétique étoit celle des anciens; l'analytique est dûe aux modernes."

today; it is in this sense, for instance, that one speaks of real and complex analysis and synthetic geometry.

Thus the development of the modern terminological hash. In philosophy, analysis continued to refer to the breaking down of certain complexes, namely concepts or cognitions, into their constituent parts, and to the search for grounds of known truths; while in mathematics, analysis became simply a branch of the science. However, the two continued to develop, as I will now describe, in very close association.

§**3**

The substantial contributions of Descartes provide a convenient place to start discussion of analysis in the modern period. Descartes follows the already established usage in distinguishing mathematical from philosophical analysis. He was aware of what the ancients called geometrical analysis, but could not believe it was the real article:

> These writers [*sc.* Pappus and Diophantus], I am inclined to believe, by a certain baneful craftiness, kept the secrets of this mathematics to themselves. Acting as many inventors are known to have done in the case of their discoveries, they have perhaps feared that their method being so very easy and simple, would, if made public, diminish, not increase the public esteem. Instead they have chosen to propound, as being the fruits of their skill, a number of sterile truths, deductively demonstrated with a great show of logical subtlety, with a view to winning an amazed admiration, thus dwelling indeed on the results obtained by way of their method, but without disclosing the method itself—a disclosure which would have completely undermined the amazement. ([22], Rule IV)

Like Viète, he thought that the true mathematical analysis had become corrupted through transmission. Mathematical analysis has a technique: it is none other than a suitably cleaned up and properly understood version of what the vulgar call algebra. As with Pappus' "Treasury", we have not just words, but a significant body of results, a detailed guide for research.

Philosophical method is not so precisely pinned down. One of its important components is analytic in nature, a matter of resolving complicated data step by step into those which are simpler. (E.g. Rule V) By seeking the simpler, Descartes reminds us, he means that we should attend to the composition of our thoughts, see which parts contribute what to them, and arrange everything in the order of cognition. (Rule VI) This movement towards the simple in cognition is thought to be at the same time a movement towards the "absolute"

> I entitle absolute whatever possesses in itself the pure and simple nature that we have under consideration, i.e. whatever is viewed as being independent, cause, simple, universal, one, equal, straight, and such like. This "absolute," this "primum," is the simplest and easiest [of apprehension], and so is of service in our further inquiries. [22, Rule VI]

As with Aristotle's essences, cognitive simples provide a solution of sorts to the question of first principles. For there are thoughts, and if these are composed of simple, absolute parts, then these are given, and given in a direct way.

The temptation to regard this as yet another empty philosophical promise was diminished considerably by what Descartes claimed as the products of his method. Prominent among these was his new mathematical analysis. It arises when the mathematics which he was able to survey is subjected to philosophical scrutiny. What is involved in all of these diverse results, he observes, is just proportions of magnitudes [9, Part 2]. Thus he cuts everything back to a minimum: instead of considering all the different varieties of magnitudes—angles, lines, planes, solids, etc.—he will consider only the simplest, straight lines. He will use an appropriate symbolism, one which will aid rather than impede the memory when things get messy. The symbolism will be kept rigorous by tying it firmly to a geometrical interpretation, the algebra of line segments and the construction of curves. Then everything opens up quite nicely:

> In this way, I should be borrowing all that is best in geometry and algebra, and should be correcting all the defects of the one by help of the other.
>
> This, I venture to assert, is what I have in fact achieved. The exact observance of these few precepts has given me such facility in unravelling all the questions dealt with by these two sciences, that in the two or three months I devoted to their examination . . . not only did I find the answer to many questions I had formerly judged very difficult, I was also in due course able, as it seemed to me, to determine, in respect even of those which I could not thus answer, by what means and to what extent an answer was yet possible. [9, Part 2]

The *Geometry* shows that there is a role for philosophical analysis in mathematics. At the same time, it also suggests that that role is only a limited and temporary one. For the philosophical part of the enterprise is over once the first principles or simples are found. And it seems that Descartes was satisfied on that score. The first principles had been set out; basic patterns of solution established; what remained was a mathematical mop-up operation, one which he left to others, as he said, so as not to deprive them of the pleasure of discovery [28, at end].

Despite its undeniable attractiveness, Descartes' account was not without its flaws. Although his new mathematical principles cast much light, it was not of the absolute sort which covered the entire mathematical universe simultaneously; instead, it seemed to decay rapidly once one moved off the initial fragment. (It was not for nothing that Descartes' waved his hands over higher degree curves at the end of his *Geometry*.) Again, a closer look suggests that the "simples" of his mathematics were not so simple after all. The simplicity of his ingenious symbolism, for example, is supported underneath by some rather clever (and complex) mathematics, notably the algebra of line segments and the invisible role of units. Evidently, the precepts guiding philosophical analysis had to be applied with some finesse. By contrast, nothing like this is tolerated in mathematical analysis, where the demands of rigour were strict. But here, intelligibility came at a high price, since algebra was tied firmly to Descartes' semantics of geometrical constructibility. One consequence was that so-called "mechanical" curves like Archimedes' spirals had to be banned from geometry.[12] And curves of this sort, already of importance in antiquity and indispensible in dynamics, were to prove of great interest in the years immediately following the publication of the *Geometry*.

§4

The simples of Descartes' philosophical analysis being, as noted, somewhat slippery, there was room for more discussion on the subject. Pascal, in an essay on the geometrical mind, had offered the following solution. In the case of concepts, he said, analysis should continue until one arrives at concepts which are clearly understood by everyone. For propositions, it should proceed until one reaches truths which are so clear that one cannot find anything more clear which might be used to prove them [23, 578-579]. Pascal's views reached a large audience through

[12] In this case, because there is no constructible link between the two movements required to generate the curve: the uniform movement of a point along a line, and the uniform rotation of the line. I.e., there is an ineliminable degree of freedom in the specification of the curve.

being summarized in the Port Royal Logic, and, like many of the doctrines set out in that work, were widely accepted as authoritative.

Leibniz was a prominent exception. In an article in the *Acta eruditorum* of 1684, after acknowledging his respect for Pascal in the lavish terms then customary, he put his finger on a thorny question. "I only wish," he wrote, "that he had defined the limits beyond which any concept or judgment is no longer even a little obscure or doubtful."[13] [20, 294] For Leibniz, indefinability of concepts occurs not because one reaches limits of clear comprehensibility, but rather because some of them are simple, having no parts. Similarly with propositions: among these some are unprovable as such, but this due to their logical form rather than their being maximally clear. Among Pascal's primitive concepts, there might be many which are still complex, and hence definable according to Leibniz. So too with principles: a proposition might well be perfectly clear and yet still not unprovable as such. Hence Leibniz's odd-sounding statement that "it is good to look for demonstrations even of axioms"[16, 101]. For here he used "axiom" in Pascal's sense, as a clear and evident proposition.

Leibniz's simple concepts and axioms form the logical foundations of science. But, like Aristotle and like Descartes, he also asked considerably more of them, requiring that they be at the same time epistemologically and metaphysically basic. The simple concepts of the understanding, he claimed on more than one occasion, are nothing other than the perfections of God.[14] Axioms, for their part, are held to be not only logically primitive, but also immediate, that is, self-evident.[15]

Leibniz's views had the potential to instigate important changes in science, and notably in mathematics. Two things were against this, however. First, Leibniz did not publish the *New Essays* due to the bad timing of Locke's death. Second, and more importantly I think, is the circumstance that—in marked contrast to Descartes and, later, Bolzano—Leibniz's best known work in mathematics by no means exemplified these general methodological views. The methodology that one could draw from the infinitesimal calculus lay much closer to the project of the general characteristic, operations like differentiation and integration being

[13] Leibniz apparently regarded this essay as a more or less definitive statement of his views, referring to it with approval as late as the *New Essays*.

[14] Cf. his remarks in [20, 293]: "Whether men will ever be able to carry out a perfect analysis of concepts, that is, to reduce their thoughts to the *first possibles* or to irreducible concepts, or (what is the same thing) to the absolute attributes of God themselves or the first causes and the final ends of things, I shall not now venture to decide."

[15] Just what sorts of truths these might be (Leibniz calls them "identities" in a idiosyncratic extended sense) is discussed in Book IV of the *New Essays*. See especially chapter ii.

justified on the basis of more general laws of symbolic forms and their transformations. Leibniz, as one might expect, had plans to unify this latter project with his general methodology. But in view of the incompleteness of his work in this area, it is quite understandable that only part of his message got through to the mathematicians who followed him.

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In part due to Leibniz's influence, mathematical analysis had by the end of the eighteenth century once again become an entirely new subject, even if it was often presented as a further development of Cartesian geometry. Three changes were of particular importance.

First, Descartes' analysis was expanded by Newton and others to include infinitary methods, notably the theory of series and the infinitesimal calculus. Among other things, this extended analysis beyond the range of Descartes' constructive semantics. Mechanical curves, for example, which had been excluded from Cartesian mathematics, could be represented with the help of series, integrals, differential equations, etc. Cartesian scruples were also violated in that the additions were not always well understood or constrained. Given the choice between a loss of intelligibility and that of a good deal of new and interesting mathematics, however, mathematicians voted with their feet for the former.

Second, the geometrical components of Cartesian analysis were expelled as "foreign elements". Analysis was instead portrayed as an autonomous science which is prior to geometry—the science of "the calculation of magnitudes and the general properties of extension" as d'Alembert described it [10, i]. Geometry was degraded to an application of the more general mathematics.

Finally, the subject matter and the conception of the foundations of analysis change: instead of curves and their properties, analysis is now concerned with analytical expressions ("functions" in Euler's sense of the word) and the rules governing their transformations. The rules, for their part, could not be justified by appealing to a geometrical semantics (though they might sometimes be verified in this way); instead, justification is sought (if at all) in purely syntactic, algebraic terms. There was a semantics: the "variable quantities" which occur in analytic expressions include, as Euler wrote, all possible complex values [17, §2]. But this semantics is Aristotelian rather than modern—it is a semantics "for the most part". The meaning of universal validity remains purely syntactic. A formula is shown to be universally valid when it has been derived using the rules of analysis:

it may nevertheless fail or give answers of a different kind for certain values of the variable quantity. Thus, for example, Euler [12] argued that since the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \tag{1}$$

may be established with the rules of analysis, we are licensed to conclude not only that

$$\frac{1}{1-\frac{1}{2}} = 2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \tag{2}$$

but also that

$$\frac{1}{1-(-1)} = \frac{1}{2} = 1 - 1 + 1 - 1 + \cdots \tag{3}$$

$$\frac{1}{1-1} = \frac{1}{0} = \infty = 1 + 1 + 1 + 1 + \cdots \tag{4}$$

$$\frac{1}{1-2} = -1 = 1 + 2 + 4 + 8 + 16 + \cdots \tag{5}$$

and

$$\frac{1}{1-3} = -\frac{1}{2} = 1 + 3 + 9 + 27 + \cdots \tag{6}$$

Bolzano would later point to such equations as claims "before which sound human reason shrinks in terror" [30, Preface, viii]. Abel, of like mind on this subject, would speak of divergent series as the work of the devil. Euler, for his part, accepted and tried to make sense of them. Somehow I find it hard to say that he didn't have any idea of what he was talking about. Part of his reasoning, to be sure, was based on a false assumption: he thought that a given infinite series could never be derived from two non-equivalent analytical expressions. But the point of the inquiry remains legitimate given his understanding of analysis: the rules of analysis license us to form the above series from the expression $\frac{1}{1-x}$. These rules make no reference to an allowable range of the variable quantity x. Hence the equation must be valid for all values, including those for which the series diverges. What seemed to follow from this—the existence of negative numbers larger than infinity—is not treated as an absurdity, but rather as an interesting consequence of the rules of analysis, on a par, perhaps, with the existence of imaginary numbers. Asking after the sum of divergent series is not very different, for Euler, from asking what the value of a^{b+ci} or of $\log(-1)$ must be. The answers to such questions are not in any way given in advance: there is no firm semantics to appeal to. Instead, one attempts to determine the result purely analytically—that is, using the available techniques for transforming analytical expressions. Often enough, this approach—especially in capable hands like Euler's—led to important extensions of analysis. Despite this success, it seems reasonable to say that there were certain points of eighteenth-century mathematical analysis which lacked sufficient clarity, something which showed itself all too often in the works of less talented mathematicians.

§5

Bolzano was one of a very small group in the early nineteenth century who were capable and well-informed in both mathematics and philosophy. Because of this, he was able to see something which escaped many of his contemporaries: namely, that mathematics, particularly its most active branch, analysis, had changed so thoroughly that the account of mathematical method which had been set out by Descartes and Pascal had lost whatever plausibility it may once have had. Basic concepts like function, continuity, differential, integral, sum of an infinite series, etc., were by no means given adequate definitions. Nor could it reasonably be claimed that they were clearly known in themselves. Similarly with proofs: even in the works of great analysts like Euler and Lagrange, these often left much to be desired, and bore scant resemblance to the earlier ideal.

As patent as the deficiencies of eighteenth-century analysis may seem in retrospect, they were all but invisible to many of Bolzano's contemporaries, and notably to philosophers. The loudest, though by no means the clearest, voices, namely those of Kant and his followers, seemed to have registered none of the changes in mathematics, and continued to propound recognizably Pascalian views. In his prize essay of 1764, for example, Kant had set out the familiar view that all mathematical concepts were either clearly known in themselves or else combinations of such concepts deliberately created by mathematicians [31]. This opinion was, in its essentials, maintained in his later writings and also by a generation or two of followers, the clarity of mathematical primitives being explained by their being constructible *a priori* in intuition. "We cannot," as he thought, "think a line without *drawing* it in thought, or a circle without *describing* it." [7, B154] Similarly for the axioms of mathematics: these are principles which, because constructible *a priori*, are immediately certain. Mathematical concepts all being clearly understood, and axioms immediately evident, there could be no room for disagreement, and as a consequence there was simply no need, no place for philosophical analysis in mathematics.

Bolzano, for his part, saw quite clearly that this was not so. Mathematics, as he well knew, was full of uncertainties and disputes concerning its concepts. This was particularly apparent in analysis: all the questions touching the infinite, the infinitely large, the infinitely small, the theory of differentials, of derivatives, integrals, of irrational numbers, and so on, were anything but definitively settled. To say that philosophical analysis had no place in mathematics was tantamount to a recommendation simply not to think about such controversial topics: a singular means, as Bolzano remarked, of removing the imperfections of the science [14, §11, Anm. 1]. At an earlier point in the development of mathematics, Bolzano allows, concentrating on foundations might well have been a waste of energy better spent enlarging the science with new discoveries [14, §11]. But mathematics had matured; and what was required for the progress of mathematics, in his view, was precisely the philosophical analysis of its basic concepts.

In an essay on the mathematical method, intended as part of the introductory matter for his *Theory of Quantity*, Bolzano presented his views on philosophical analysis in conjunction with his usual survey of the opinions of others. He begins with the analysis of conceptual content. The concepts found in our consciousness are said to be *distinct*, he writes, if we are able to indicate the parts of which they are composed, and the order in which they are put together. (Bolzano, 1981, §10) In a rigorous presentation, he writes, one should endeavor, insofar as possible, to indicate the parts of the propositions and concepts of the science, that is, to make them as distinct as possible. This is especially important in that it better enables us to indicate what he calls the objective connection between truths.

A maximum of distinctness will be obtained, he thinks, when we define a concept in terms of simple components. Thus for him, as for Leibniz, conceptual analysis has simple concepts as its terminus. By contrast, however, these simple concepts are for Bolzano merely semantic primitives, relieved of the heavy epistemological and metaphysical burdens Leibniz had given them. Indeed, the designation of a concept as simple is, as I will discuss shortly, of necessity a partly conjectural business. If we couple with this the circumstance that the simple concepts which Bolzano turned up tended to be such things as "point", "have", "something", "should", "not", we will not be surprised that he was not tempted to regard these as absolutes, divine perfections, etc.

Bolzano's adherence to received terminology is quite misleading, and it must be emphasized that his analysis was by no means the traditional one of the contents of the mind. For one thing, the subjective concepts associated with given words may differ widely between individuals, so there will not be, in general, a unique thought content for analysis to address. In addition, the result of conceptual analysis, namely, a definiens for a certain expression, may not correspond to anyone's, let alone everyone's, subjective idea. When Bolzano, after careful consideration, presented definitions of 1-, 2- and 3-dimensional spatial objects as certain kinds of point-sets, he by no means claimed to be reporting the contents of anyone's thoughts. With these definitions, as with many others Bolzano developed, it is quite likely that no one had even entertained them before him.

What individuals think is of importance, to be sure; but individual thoughts are not the matter of philosophical analysis in the way that a dogfish is of a dissection. The parts of concepts are not given, waiting to be discovered. Instead, they have to be conjectured. One forms, namely, various combinations of other concepts and tests them against the concept one seeks to analyse. A proposed definition is adequate not if it tells people what they have been thinking, but rather if it does justice to usage.

> One might, for example, want to dispute forever about whether we have given the correct concept of the expression "extended spatial object" when we define it as a spatial object of such a kind "every one of whose points, at every distance no matter how small, has certain neighbors"—if we are in a position to derive from this concept all the properties that one knows of extended spatial objects, then it will be shown that our concept, if not identical with the customary one, is at least equivalent to it, and one will have cause to be satisfied with it.[16][14, §11]

This account of conceptual analysis has the usual two aspects for Bolzano. On the objective level, where one deals with concepts and propositions in themselves, the notion of conceptual analysis moves towards formal criteria of definability. A necessary condition for a complex concept to serve as a definition of a given concept will be Leibniz's substitutability *salva veritate*: i.e. the truth values of all propositions containing the given concept remain unchanged when the complex concept is substituted uniformly for it. (Other criteria, e.g. for eliminating the possibility of circular definitions, must obviously also be met.)

[16]"So möchte man z. B. immer noch darüber streiten wollen, ob wir den rechten Begriff des Ausdrückes 'ein ausgedehntes Raumding' angeben, wenn wir dasselbe als ein Raumding von solcher Art erklären, 'dessen jeder Punkt für jede noch so klein Entfernung gewisse nachbarn hat:'—wenn wir im Stande sind, alle Beschaffenheiten, die man von ausgedehnten Raumdingen kennt, aus diesem Begriffe herzuleiten : so wird erwiesen seyn, daß unser Begriff, wo nicht der nähmlich mit dem gewöhnlichen, doch ein ihm gleichgeltender sey, und man wird Ursache haben, mit unserer Erklärung zufrieden zu seyn."

On the subjective level, where analysis is put to work, things are somewhat more complicated.[17] For here one has not concepts, but words and thoughts, and, in the case of most sciences, a literature and established patterns of usage. To analyse a concept one must first be well informed on how it is used. Attempted definitions cannot, as in the objective case, cover all possibilities; instead, one takes a stab at setting out a list of plausible candidates. These are tested not against all propositions which can be formed with the concept in question, but against those which, upon reflection and after some selection, are judged to be relevant to its correct use. If a candidate proves equivalent over this set of propositions, we are justified in provisionally accepting it as a successful analysis. If after repeated trials, on the other hand, no complex concept is found which passes the test, we are justified in provisionally accepting the concept at hand as simple. In such cases, however, we should tell our readers why and how we have come to this decision, explaining how various attempts have come up short.[18]

Now the set of propositions considered in such attempts at analysis is almost certain to leave the problem of definition underdetermined, something which can even lead to surprises. One might, for instance, frame a definition of continuous function which is found to be satisfactory in that it allows us to derive all the generally acknowledged properties of continuous functions: they have no gaps, jumps, etc. But is a continuous function so defined necessarily differentiable everywhere except at an isolated set of points, as many (including Bolzano himself at one time) assumed? Perhaps—and, in the case the definition Bolzano settled on, provably—not.

Thus there is a certain freedom inherent in foundational inquiries, some room for creativity. Gone, at the same time–although it is not clear how far Bolzano was aware of this—is the temptation to suppose that a definitive foundation for mathematics will ever be attained. For certain well-delimited branches, like straightedge and compass geometry, perhaps. But one of the effects of foundational work, as Bolzano noted, is to lead us to new theorems which were previously unattainable, something which may put us in a position to pose our questions differently. Once, for example, one has a definition of integration, it becomes possible to investigate its properties, its algebraic structure, etc. At this point, one is able to ask whether it might not be possible or worthwhile to attempt to define the concept differently, in order, for example, to render a larger class of functions integrable, or to ob-

[17] Cf. Bolzano's discussion on how to determine the parts of a subjective representation in the *Wissenschaftslehre*, §350.

[18] [14, §11]; cf. [32, §23], where Bolzano explains his failure to define "proposition", for an example of the sort of discussion he had in mind.

tain better closure properties, etc. It is not clear in advance that iterations of this process will lead one to a fixed point.

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The deductive order of science, similarly, is subjected to philosophical analysis. Again following Leibniz, Bolzano thought that the axioms, or basic truths, must be truths which are unprovable as such. But he saw no good reason to suppose that this logical property must also be invariably accompanied by self-evidence. As he remarked in 1810, axioms may be less evident than the consequences which follow from them; indeed, they may even appear false as long as one has not fully appreciated their role in supporting accepted truths [27, II, §21]. Obviousness not only varies with experience and from one individual to the next, it is also apparently fully compatible with falsity. Thus early on, with reason, Bolzano had dismissed it from his methodology.

Instead, he attempted to set up formal criteria for axioms and for that certain kind of deductive dependence which he called the objective connection between truths. Bolzano was never entirely settled in his opinion on the nature of this objective order, but he did at least sketch some of the considerations which guided his work in foundations. A central element of these criteria in his later writings is the notion of *exact deducibility*, which combines features of Bolzano's general concept of deducibility with those of formal independence. A proposition M is said to be exactly deducible from the propositions $A, B, C, \ldots$ relative to designated occurrences of certain variable parts $i, j, k, \ldots$ iff every uniform substitution of ideas $i', j', k', \ldots$ for $i, j, k, \ldots$ which makes all of $A, B, C, \ldots$ true also makes M true; and the same does not hold for any proper part of the set of propositions $A, B, C, \ldots$.[19] (Bolzano proves in the *Wissenschaftslehre* that if M is exactly deducible from $A, B, C, \ldots$ then none of $A, B, C, \ldots$ is deducible from the remaining propositions in the list; also that neither M nor any of $A, B, C, \ldots$ can be analytic [32, §155, 27].) The notion of exact deducibility eliminates a number of redundant and trivial inferences, but was not completely satisfactory for Bolzano's purposes. Hence, he added a criterion of simplicity: the grounds of a truth should be, he thought, no more complex than their consequence, in the sense that the grounds, taken collectively, should contain no simple concepts which do not also occur in the consequence [32, §221, 7].

[19] [14, §8]; cf. [32, §155, 26] for an earlier, somewhat different formulation.

These criteria do not completely pin down the relation of ground and consequence, but they are already sufficient to give a good deal of shape to the analysis of the objective dependence of truths. A search for the grounds of a given proposition will begin with a determination of which concepts it contains. The grounds which one adduces for the proposition should be other propositions involving the concepts contained in it. Then the problem becomes one of reverse mathematics: to find a set of premises using only these concepts from which the given proposition is exactly deducible. At a certain point, one will, Bolzano thinks, arrive at propositions which, due to their degree of simplicity and their deductive relations, appear as grounds but never as consequences of the other propositions of a science; and these are axioms. By their nature, axioms cannot be proved. They must nevertheless be justified; not, however, by their self-evidence; rather, because after repeated trials of various possibilities, a given set seems to do the job of supporting the deductive structure of a science best.[20]

To ground a proposition, for Bolzano—to answer Aristotle's question, *why*?—is, as Jan Sebestik has written, "to integrate it into the system of scientific truths ordered by the ground-consequence relation" [24, 94]. The target of analysis is now the elaboration of a conceptual and deductive structure: the specification of primitive, or simple concepts, of axioms, and the presentation of the definitional and deductive relations linking these primitive elements with the proposition in question. Thus presented, philosophical analysis is truly global, a searching for ultimate grounds; and we can readily see why Bolzano felt compelled to develop theories of collections, of integers, and of real numbers (among others) in order to ground the propositions of real function theory. What emerged from these inquiries was yet another version of mathematical analysis, modern real analysis more or less as we still find it today.

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Consider now a small sample of Bolzano's mathematical work where this new conception of philosophical analysis had been put to work: his proof of the proposition that a real-valued function which is continuous on a closed interval is bounded there [4, I, §§19-21]. An inspection of the statement of this theorem indicates that we have to do with the concepts of real number, function, boundedness, and continuity on an interval. The grounding of the proposition should therefore make use

[20] Cf. [27, II, §21].

of propositions involving these concepts. These propositions, in their turn, must be preceded by adequate definitions of the concepts in question. Accordingly, we find that Bolzano has constructed an arithmetical theory of real (or, as he calls them, "measurable") numbers, has given a quite general definition of function, and a thoroughly modern definition of pointwise continuity on an interval. He has also (he says, although the proof has not been found in his papers) proved an important statement of the completeness of the real numbers, the so-called Bolzano-Weierstrass theorem: if S is a infinite set of real numbers all of whose members lie within a closed interval $[a, b]$, then there exists a real number c in $[a, b]$ such that, for any real number $j > 0$, however small, infinitely many elements of S lie within $(c - j, c)$ or within $(c, c + j)$.

The proof runs as follows: Suppose a function f is unbounded on the closed interval $[a, b]$. Then for $n = 1, 2, 3, \ldots$ there exist $x_1, x_2, x_3, \ldots$ in $[a, b]$ such that $|f(x_n)| \geq n$. By the Bolzano-Weierstrass theorem, there will be a point c in $[a, b]$ every neighborhood of which contains infinitely many of the x_i. It follows that f is unbounded in every neighborhood of c and, as an easy consequence, that f is not continuous at c. It has thus been shown that if f is unbounded on $[a, b]$ it is not continuous on the interval; and the result sought follows by contraposition.

Once we have set out a clear statement of the theorem, and adequate definitions of the concepts contained in it, we can see the role that each of these—boundedness, continuity on a closed interval, real number—has to play. We can see, for instance, not only *that*, but *why* the condition of a closed interval is necessary. Constructing proofs in this sense is not just a mathematical but also a philosophical activity.

In his extensive work on function theory, Bolzano makes the occasional mistake, but never puts a foot wrong with respect to his basic orientation. It is clear that he has a very definite idea of what is involved in the new mathematics. It is thus entirely natural to describe Bolzano as one of the founders of modern real analysis. But it is also important to recognize in his work what Russell later saw in the mathematics of Weierstrass and Cantor: the clear trace of a new kind of philosophical analysis.

§6

It has often been said that Bolzano had little impact on the historical development of philosophy. Husserl, for example, thought that Bolzano's influence, even in Austria, had been very limited. In support of this opinion, he cited the sales of Bolzano's greatest philosophical work, the *Wissenschaftslehre*, which were, indeed, less than brisk [25].[21] Even had no one read any of Bolzano's properly philosophical writings, however, the question of influence would still remain open. For an important avenue, namely mathematics, would have been passed over in silence. The early analytic philosophers mentioned by Russell, Weierstrass and Cantor, were certainly aware of Bolzano's mathematics: the 1817 *Purely analytic proof*, which despite its brevity contains many crucial elements for the rigorous foundations of analysis, was carefully studied by Weierstrass and his circle. The *Paradoxes of the Infinite* found an equally select audience. Questions of influence are notoriously vexed, especially in mathematics, where similar ideas are often arrived at independently. But we can still say with some confidence that Bolzano had an important impact on the history of mathematical analysis.

Do not those developments in methodology, in the understanding of foundations, so clearly exemplified in Bolzano's work, also belong to the history of philosophy? Bolzano certainly thought so: most of his mathematical research he thought of as being philosophical at the same time. But most histories of philosophy seem to suggest otherwise: in discussions of the nineteenth century, they find much room for Fichte, Hegel, even Schopenhauer and Nietzsche, but is is rare indeed to find even a mention of, say, Weierstrass. Russell, it seems to me, showed sounder judgment in recognizing mathematics as an important vehicle for philosophical content. His error was only one of oversight: not having a detailed acquaintance with Bolzano's work, he did not know that, in many of the cases he thought most important, the philosophy which he found in nineteenth-century mathematics had been quite carefully put there in the first place. But his oversight seems infinitesimal in comparison with that committed by those historians who overlooked mathematics altogether.

[21] Bolzano views on the subject of influence, given in his autobiography [18, 80], are worth quoting here. After relating some of the difficulties he had had in communicating his ideas due to the harassment of the Austrian authorities (e.g., the dismissal from his university post, the order forbidding him to preach, manuscripts submitted to the censor neither approved nor returned, etc.), he writes: "Was mir bey solchen Betrachtungen tröstet, ist der Spruch Jesu : Wer euch den Leib gegeben hat, wird er nicht auch die Kleidung euch geben ? Dieß wende ich nähmlich auf meine Verhältnisse so an. Wenn die Begriffe, die dich Gott finden ließ, in der That Wahrheit und nützliche Wahrheit enthalten, so hat er die das Mehrere (den Leib) gegeben ; kein Zweifel also, daß er zur rechten Zeit dir auch die Gelegenheit zu ihrer Ausbreitung, die etwas viel Minderes (das Kleid zum Leibe) ist, geben werde."

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